Contract No. DA-04-495-Ord-19

Army Ordnance Department

Sponsored by Army Ordnance and Air Force

MEMORANDUM NO. 9

SHOCK WAVE EFFECTS ON THE LAMINAR SKIN FRICTION OF AN INSULATED FLAT PLATE AT HYPERSONIC SPEEDS

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July 1, 1952

ABSTRACT

An approximate theory on the phenomena of interaction between the shock wave and the laminar boundary layer on an insulated flat plate at hypersonic speeds has been formulated. Results on the rate of growth of the boundary layer thickness and the rate of decay of the shock wave strength have been found that hold for $M_1^{2(2+\omega)} R_1^{-1} = O(1)$. A new set of formulae for the average skin friction coefficient, C_F, over an insulated flat plate at hypersonic speeds has been obtained. Calculations on the basis of the new C_F formulae yield the data shown in Figs. 5 and 6. Contrary to the conventional theory which predicts a steady decrease in C_F as M_1 increases, the present results indicate that C_F may increase with M_1 at hypersonic Mach numbers.

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NOMENCLATURE

L	Chord of the flat plate
x_o	Interaction distance
C_F	Average skin friction coefficient
M_1	Free stream Mach number
x, y	Cartesian coordinates (Fig. 1)
u, v	Velocity components in x,y directions respectively
p	pressure
ρ	density
T	Absolute temperature
μ	Coefficient of viscosity
k	Coefficient of conductivity
R	gas constant
c_p	Specific heat at constant pressure
ω	Constant index of the power law of viscosity
$R_{1x} = \frac{\rho_1 u_1 x}{\mu_1}$	Free stream Reynolds number based on the distance x from the leading edge
$R_1 = \frac{\rho_1 u_1 L}{\mu_1}$	Free stream Reynolds number based on the chord of the flat plate
δ	Boundary layer thickness
$P_r = \frac{\mu c_p}{k}$	Prandtl number
$\tau_w = \mu_w \left(\frac{\partial u}{\partial y}\right)_w$	Laminar skin friction of the flat plate

Subscripts "1" and "2" refer to the conditions pertaining to the free stream and the outer boundary of the viscous layer respectively. Subscript "w" refers to the conditions on the flat plate surface.

I. INTRODUCTION

In recent years, a considerable amount of information concerning the skin friction of a two-dimensional thin flat plate moving edgewise through the air at high speeds has been obtained through the study of compressible boundary layer equations (Ref. 1). The usual basic assumption is that viscous effects are confined to a narrow region close to the plate surface. The flow field outside this narrow region is regarded as non-viscous and not effected by the presence of the boundary layer. However, when the plate is moving at supersonic speeds, disturbances due to the viscous effects in the boundary layer can drastically influence the outside flow field (Ref. 2). Thus, the retarded flow in the boundary layer causes the streamlines to turn slightly away from the plate surface and this results in a shock wave extending from the plate into the free stream. This shock wave, moreover, decays in its strength as it intersects successive expansion waves originating from the boundary layer. There is, therefore, a close relationship between the rate of growth of the boundary layer thickness in the viscous flow region and the rate of decay of the shock wave strength in the outside flow field. In the lower supersonic speed range, the boundary layer and the shock wave are so far apart that their interaction effects can probably be neglected as in the existing theories (Ref. 1). The situation is quite different when the hypersonic speed range is approached. One may define the hypersonic flow regime as such that $M_1 \frac{v_2}{u_2} = O(1)$ (Ref. 3). By the familiar argument, the boundary layer thickness must be such that $\frac{\delta}{x} = O(\frac{v_2}{u_2}) = O(\frac{1}{M_1})$ (Cf. Appendix). Therefore, in the hypersonic speed range, the shock wave, which has a wave angle of the order of the quantity $1/M_1$, is so close to the plate surface that the entire region between the shock wave and the plate surface should be considered as a viscous flow layer. Consequently, for the determination of the shock wave characteristics, viscous effects must be considered (Ref. 4); and for the estimation of the plate skin friction, due care must be exercised to account for the shock wave effects. In the present paper, an approximate theory of the shock wave effects on the hypersonic laminar friction on an insulated flat plate will be presented.

II. BASIC EQUATIONS AND ASSUMPTIONS

The thickness of the shock wave is probably of the order of the molecular mean free path (Refs. 5 and 6). Assuming a continuum flow, one can regard the shock wave as being infinitesimally thin. The flow variables in the region bounded by the shock wave and the plate surface are assumed to satisfy the compressible boundary layer equations:

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{dP_2}{dx} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) \tag{1}$$

$$\frac{\partial}{\partial x}\rho u + \frac{\partial}{\partial y}\rho v = 0 \tag{2}$$

$$\rho u \frac{\partial}{\partial x}(c_p T) + \rho v \frac{\partial}{\partial y}(c_p T) = u\frac{dP_2}{dx} + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2 \tag{3}$$

The order of magnitude arguments which are responsible for the derivation of the above equations can be carried through in hypersonic flow problems. (Cf. Appendix). However, these arguments can fail near the base of the shock wave (Ref. 7). In the present analysis, the shock wave is assumed to start from the plate leading edge (Fig. 1). Therefore, the above equations do not hold in the vicinity of the origin, $0 \leq x < \epsilon$. The flat plate possesses , in a certain sense, a fictious curvature due to the presence of the viscous layer. It is because of the fictious curvature effects that the terms containing dp_2/dx are retained in Eqs. 1 and 3.

The problem is then to seek the solution of the boundary layer equations (1-3) with the appropriate boundary conditions. The conditions of no slip and heat insulation are assumed valid at the surface of the flat plate. At the outer boundary of the viscous layer, besides the usual conditions, the pressure jump condition across an oblique shock wave has to be satisfied. Thus, one has the following conditions:

$$\text{At } y = 0, \qquad u = v = 0, \qquad \left(\frac{\partial T}{\partial y}\right)_w = 0 \tag{4}$$

$$\text{At } y = \delta, \qquad u = u_2, \quad \left(\frac{\partial u}{\partial y}\right)_\delta = \left(\frac{\partial^2 u}{\partial y^2}\right)_\delta = 0, \quad \frac{p_2}{p_1} = \frac{2\gamma}{\gamma+1}\left(M_1 \frac{d\delta}{dx}\right)^2 - \frac{\gamma-1}{\gamma+1} \tag{5}$$

It may happen that the rate of decay of the shock wave strength is such that at $x = x_0$ the shock wave already degenerates into a Mach wave. When this happens, a new phase in the interaction between the outside flow and the boundary layer begins such that the entire problem has to be formulated differently.* The present problem is therefore to find the solution satisfying Eqs. 1-5 in the domain $0 \leq y \leq \delta$, $\epsilon < x \leq x_0$.

In the present analysis, the following simplifying assumptions are made: (1) the perfect gas law $p = R\rho T$ is assumed valid; (2) the air viscosity coefficient is a function of temperature only, the simple law

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* One may assume that for $x > x_0$, the boundary layer growth is so slow that the nonlinear interactions between the shock and the boundary layer appear only as higher order effects and are negligible. x_0 can therefore be designated as the "interaction distance". For $x > x_0$, the ordinary boundary layer theory is assumed valid.

$\mu \propto T^{\omega}$ is assumed; (3) the specific heat at constant pressure is assumed constant and the Prandtl number is assumed unity. Then, the particular energy integral

$$c_p T + \tfrac{1}{2} u^2 = \text{constant} \tag{6}$$

which satisfies the conditions in Eq. 4, reduces Eq. 3 identically to Eq. 1. Integrating Eqs. 1 and 2 across the boundary layer and combining the results yield the familiar momentum integral relation

$$\frac{\tau_w}{\rho_1 u_1^2} = \frac{d}{dx}\frac{\rho_2 u_2^2}{\rho_1 u_1^2}\int_0^\delta \frac{\rho u}{\rho_2 u_2}\left(1-\frac{u}{u_2}\right)dy - \frac{\rho_2 u_2}{\rho_1 u_1}\frac{d}{dx}\left(\frac{u_2}{u_1}\right)\int_0^\delta \frac{\rho u}{\rho_2 u_2}dy - \frac{p_1}{\rho_1 u_1^2}\frac{d}{dx}\left(\frac{p_2}{p_1}\right)\int_0^\delta dy \tag{7}$$

By Eq. 1 and the boundary conditions in Eq. 5, we have

$$\rho_2 u_2 \frac{du_2}{dx} = -\frac{dp_2}{dx} \tag{8}$$

which on substitution into Eq. 7 transforms the latter equation into the following form:

$$\frac{\tau_w}{\rho_1 u_1^2} = \frac{d}{dx}\frac{\rho_2 u_2^2}{\rho_1 u_1^2}\int_0^\delta \frac{\rho u}{\rho_2 u_2}\left(1-\frac{u}{u_2}\right)dy + \frac{p_1}{\rho_1 u_1^2}\frac{d}{dx}\left(\frac{p_2}{p_1}\right)\int_0^\delta \left(\frac{\rho u}{\rho_2 u_2}-1\right)dy \tag{7a}$$

By Eq. 6, one can show that

$$\frac{T}{T_2} = 1 + \frac{\gamma-1}{2}M_1^2\frac{T_1}{T_2}\left(\frac{u_2}{u_1}\right)^2\left(1-\frac{u^2}{u_2^2}\right)$$

which for $M_1 >> 1$ can be approximated in the layer $0 \leq y < \delta(1-\frac{\Delta_1}{\delta})$ as

$$\frac{T}{T_2} \simeq \frac{\gamma-1}{2}M_1^2\frac{T_1}{T_2}\left(\frac{u_2}{u_1}\right)^2\left(1-\frac{u^2}{u_2^2}\right) \tag{9}$$

Inside the boundary layer, one has $\frac{\rho}{\rho_2} = \frac{T_2}{T}$ for x = constant. Therefore, the momentum integral relation in Eq. 7a can be rewritten as

$$\frac{\tau_w}{\rho_1 u_1^2} = \frac{2}{(\gamma-1)M_1^2}\frac{d}{dx}\left[\frac{p_2}{p_1}\int_0^{\delta(1-\frac{\Delta_1}{\delta})}\frac{\frac{u}{u_2}}{1+\frac{u}{u_2}}dy\right] + \frac{p_1}{\rho_1 u_1^2}\frac{d}{dx}\left(\frac{p_2}{p_1}\right)\int_0^{\delta(1-\frac{\Delta_1}{\delta})}\left[\frac{2}{(\gamma-1)M_1^2}\frac{T_2}{T_1}\left(\frac{u_1}{u_2}\right)^2\frac{\frac{u}{u_2}}{1-\left(\frac{u}{u_2}\right)^2}-1\right]dy$$

$$+\frac{d}{dx}\left[\frac{\rho_2 u_2^2}{\rho_1 u_1^2}\int_{\delta(1-\frac{\Delta_1}{\delta})}^{\delta}\frac{\frac{u}{u_2}\left(1-\frac{u}{u_2}\right)}{1+\frac{\gamma-1}{2}M_1^2\frac{T_1}{T_2}\left(\frac{u_2}{u_1}\right)^2\left(1-\frac{u^2}{u_2^2}\right)}dy\right] \tag{10}$$

$$+\frac{p_1}{\rho_1 u_1^2}\frac{d}{dx}\left(\frac{p_2}{p_1}\right)\left[\int_{\delta(1-\frac{\Delta_1}{\delta})}^{\delta}\frac{\frac{u}{u_2}}{1+\frac{\gamma-1}{2}M_1^2\frac{T_1}{T_2}\left(\frac{u_2}{u_1}\right)^2\left(1-\frac{u^2}{u_2^2}\right)}dy - \Delta_1\right]$$

For $M_1 >> 1$, $\frac{\Delta_1}{\delta} << 1$. Eq. 10 can finally be approximated as follows:

$$\frac{\tau_w}{\rho_1 u_1^2} = \frac{2}{(\gamma-1)M_1^2}\frac{d}{dx}\left[\frac{p_2}{p_1}\int_0^\delta \frac{\frac{u}{u_2}}{1+\frac{u}{u_2}}dy\right] - \frac{\delta}{\gamma M_1^2}\frac{d}{dx}\left(\frac{p_2}{p_1}\right) \tag{11}$$

The above equation was first used by Shen (Ref. 4) in his research on the viscous effects on the shock wave shape over an insulated wedge at hypersonic speeds.

III. NONLINEAR INTERACTIONS BETWEEN THE SHOCK WAVE AND THE BOUNDARY LAYER

The momentum integral equation, Eq. 11, as it stands, can tell us nothing about the values of δ , τ_w or p_2/p_1, or their variation with x. To obtain these values it is necessary to make some more or less plausible assumptions concerning the distribution of the velocity u, satisfying as many of the boundary conditions as is convenient. The boundary conditions in Eq. 4 can be rewritten as follows:

$$\text{At } y = 0, \qquad u = 0, \qquad \frac{dp_2}{dx} = \left[\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)\right]_w = \mu_w\left(\frac{\partial^2 u}{\partial y^2}\right)_w \tag{4a}*$$

where the second condition is derived from Eq. 1. The given boundary conditions except the one pertaining to the pressure jump condition in Eq. 5 are satisfied by

$$\frac{u}{u_2} = f(y^*;\Lambda) \tag{12}$$

where

$$y^* = \frac{y}{\delta} \tag{13}$$

$$\text{if} \quad f(1) = 1, \quad f'(1) = f''(1) = 0, \quad f(0) = 0, \; f''(0) = -\Lambda = \frac{\delta^2}{\mu_w u_2}\frac{dp_2}{dx} \tag{14}$$

Following Pohlhausen, (Refs. 8 and 9), we shall take f to be a polynomial of fourth degree as follows:

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* $\left(\frac{\partial\mu}{\partial y}\right)_w \propto \omega T_w^{\omega-1}\left(\frac{\partial T}{\partial y}\right)_w = 0$

$$f = 2y^* - 2y^{*3} + y^{*4} + \frac{\Lambda(x)}{6} y^*(1-y^*)^3 \qquad (15)^*$$

Thus, according to this approximate method of solution of Eq. 11, the velocity distributions across the sections of the viscous layer, when expressed nondimensionally, are all represented by a one parameter family of curves. Moreover, the velocity distribution at any section depends on the function $\frac{dp_2}{dx}$ at that section only, being affected by the state of conditions upstream only in so far as this affects δ.

By Eq. 15, the momentum integral relation in Eq. 11 becomes simply

$$\frac{\mu_w\left(2+\frac{\Lambda}{6}\right)}{\rho_1 u_1 \delta} = \frac{2}{(\gamma-1)M_1^2}\frac{d}{dx}\left[\frac{p_2}{p_1}\delta\left(1-F(\Lambda)\right)\right] - \frac{\delta}{\gamma M_1^2}\frac{d}{dx}\frac{p_2}{p_1} \qquad (16)$$

where

$$F(\Lambda) = \int_0^1 \frac{dy^*}{1+f} \,. \qquad (17)$$

Rearranging the terms in Eq. 16 yields

$$\frac{p_2}{p_1}\frac{d\delta^2}{dx} - \frac{2F'(\Lambda)}{1-F(\Lambda)}\frac{d\Lambda}{dx}\frac{p_2}{p_1}\delta^2 = \frac{\left(2+\frac{\Lambda}{6}\right)\mu_w(\gamma-1)M_1^2}{\rho_1 u_1\left[1-F(\Lambda)\right]} - \left(2 - \frac{\gamma-1}{\gamma\left[1-F(\Lambda)\right]}\right)\delta^2\frac{d}{dx}\frac{p_2}{p_1} \qquad (16a)$$

where $F'(\Lambda) = \frac{dF}{d\Lambda}$. The last condition in Eq. 14 states that

$$\delta^2\frac{d}{dx}\frac{p_2}{p_1} = -\frac{\Lambda\mu_w u_2}{p_1} \simeq -\frac{\Lambda\mu_w u_1}{p_1} \qquad (18)^{**}$$

Adding this last quantity to both sides of Eq. 16a and reducing yields

$$\frac{d}{dx}\left(\frac{p_2}{p_1}\delta^2\right) - \frac{2F'(\Lambda)}{1-F(\Lambda)}\frac{d\Lambda}{dx}\left(\frac{p_2}{p_1}\delta^2\right) = M_1^2\frac{\mu_w}{\rho_1 u_1}\left[\frac{\gamma-1}{1-F(\Lambda)}\left(2-\frac{5\Lambda}{6}\right)+\gamma\Lambda\right] \qquad (16b)$$

Eq. 16b is a linear ordinary differential equation of the first order which can be integrated into the following result:

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* Shen (Ref. 4) used a velocity profile linear in y*. He could satisfy only two of the conditions in Eq. 14, viz., f(1) = 1, f(0) = 0.

** u_1 and u_2 are related across the shock wave by the relation $\frac{u_2}{u_1} = \left(1+\frac{v_2}{u_2}\frac{d\delta}{dx}\right)^{-1} = 1 - O\left(\frac{1}{M_1^2}\right)$ because $\frac{v_2}{u_2} = O\left(\frac{1}{M_1}\right)$ and $\frac{d\delta}{dx} = O\left(\frac{1}{M_1}\right)$. For hypersonic flow, one can take $\frac{u_2}{u_1} \simeq 1$.

$$\frac{p_2}{p_1}\delta^2 = \frac{M_1^2 \frac{\mu_w}{\rho_1 u_1}}{[1-F(\Lambda)]^2}\int_0^x [1-F(\Lambda)]^2\left[\frac{(\gamma-1)(2-\frac{5\Lambda}{6})}{1-F(\Lambda)}+\gamma\Lambda\right]dx \tag{19}$$

The quantity p_2/p_1 must, moreover, satisfy the pressure jump condition mentioned in Eq. 5. Therefore, we have the following equation:

$$\frac{\gamma}{2(\gamma+1)}\left(\frac{d\eta}{dx}\right)^2 - \frac{\gamma-1}{\gamma+1}\frac{\eta}{M_1^2} = \frac{\frac{\mu_w}{\rho_1 u_1}}{[1-F(\Lambda)]^2}\int_0^x [1-F(\Lambda)]^2\left[\frac{(\gamma-1)(2-\frac{5\Lambda}{6})}{1-F(\Lambda)}+\gamma\Lambda\right]dx \tag{20}$$

where

$$\eta = \delta^2. \tag{21}$$

On the other hand, the pressure jump condition may be substituted in Eq. 18 to yield another equation

$$\Lambda = -\frac{\gamma}{\gamma+1}\frac{M_1^2 \rho_1}{\mu_w u_1}\frac{d\eta}{dx}\left[\frac{d^2\eta}{dx^2} - \frac{1}{2\eta}\left(\frac{d\eta}{dx}\right)^2\right] \tag{22}$$

Eqs. 20 and 22 form a pair of equations which define Λ and η. The problem is to find Λ and η satisfying Eqs. 20 and 22 simultaneously. η is the boundary layer thickness function while Λ is essentially the shock wave pressure function. Eqs. 20 and 22 show the interdependence of the growth of the boundary layer and the decay of the shock wave. The non-linear interaction characteristics are very clearly exhibited.

IV. APPROXIMATE SOLUTIONS OF THE BASIC EQUATIONS

To deal with the dual relations in Eqs. 20 and 22, exactly as they stand, one may eliminate Λ between them to obtain one equation defining η. The resulting equation is, however, very nonlinear, a rigorous treatment of which would be extremely tedious and difficult. For hypersonic flow, $M_1 >> 1$, we shall consider the following approximate equations instead:

$$\frac{\gamma}{2(\gamma+1)}\left(\frac{d\eta}{dx}\right)^2 = \frac{\frac{\mu_w}{\rho_1 u_1}}{[1-F(\Lambda)]^2}\int_0^x [1-F(\Lambda)]^2\left[\frac{(\gamma-1)(2-\frac{5\Lambda}{6})}{1-F(\Lambda)}+\gamma\Lambda\right]dx \tag{20a}$$

$$\Lambda = -\frac{\gamma}{\gamma+1}\,\frac{M_1^2 p_1}{\mu_w u_1}\,\frac{d\eta}{dx}\left[\frac{d^2\eta}{dx^2}-\frac{1}{2\eta}\left(\frac{d\eta}{dx}\right)^2\right] \tag{22}$$

The omission of the term $-\frac{\gamma-1}{\gamma+1}\frac{\eta}{M_1^2}$ can be justified a posteriori in a certain limited range of x's (Cf. next Section). Notice that some of the nonlinearity properties of Eqs. 20 and 22 have been preserved; the treatment of the approximate system should provide an initial step towards the understanding of the exact system. From Eq. 20a, η is easily obtained as follows:

$$\eta = \left[\frac{2(\gamma+1)}{\gamma}\frac{\mu_w}{\rho_1 u_1}\right]^{\frac{1}{2}}\int_0^x [1-F(\Lambda)]^{-1}\left\{\int_0^{x_1}[1-F(\Lambda)]^2\left[\frac{(\gamma-1)(2-\frac{5\Lambda}{6})}{1-F(\Lambda)}+\gamma\Lambda\right]dx_2\right\}^{\frac{1}{2}}dx_1 \tag{23}$$

It can be verified that $\Lambda = \Lambda_o$ = constant satisfies Eqs. 22 and 23 simultaneously. The constant Λ_o must satisfy the following equation:

$$\gamma\Lambda_o = \frac{(\gamma-1)(2-\frac{5\Lambda_o}{6})}{1-F(\Lambda_o)} \tag{24}$$

Eq. 23 can be then evaluated as follows:

$$\eta_o = \frac{4}{3}\left[(\gamma+1)\,\frac{\mu_w\Lambda_o}{\rho_1 u_1}\right]^{\frac{1}{2}} x^{3/2} \tag{25}$$

In terms of the approximate boundary layer thickness δ_o, we have

$$\frac{\delta_o}{x} = \frac{2}{3^{1/2}}\left[(\gamma+1)\,\frac{\mu_w}{\mu_1}\left(\frac{\mu_1}{\rho_1 u_1 x}\right)\Lambda_o\right]^{1/4} \tag{26}$$

By Eq. 9, $\frac{\mu_w}{\mu_1}$ can be expressed as

$$\frac{\mu_w}{\mu_1} = \frac{\mu_w}{\mu_2}\frac{\mu_2}{\mu_1} = \left(\frac{T_w}{T_2}\frac{T_2}{T_1}\right)^{\omega} \simeq \left(\frac{\gamma-1}{2}M_1^2\right)^{\omega}$$

Thus, the boundary layer growth follows the following approximate law:

$$\frac{\delta_o}{x} = \frac{2}{3^{1/2}}(\gamma+1)^{1/4}\left(\frac{\gamma-1}{2}\right)^{\omega/4}\Lambda_o^{1/4}M_1^{\omega/2}R_{1x}^{-1/4} \tag{26a}$$

It can easily be shown that

$$\frac{d\delta_o}{dx} = \frac{3}{4}\frac{\delta_o}{x} = \frac{3^{1/2}}{2}(\gamma+1)^{1/4}\left(\frac{\gamma-1}{2}\right)^{\omega/4}\Lambda_o^{1/4}M_1^{\omega/2}R_{1x}^{-1/4} \tag{27}$$

In Eq. 27, $\frac{d\delta_o}{dx}$ is the approximate slope of the shock wave which, as assumed, forms the outer boundary of the viscous layer. The result indicates that as x increases, the slope of the shock wave decreases according to $x^{-\frac{1}{4}}$. Therefore, the shock wave will decay approximately in the manner prescribed by Eq. 27. At $x = \hat{x}_0$, the shock wave will eventually be parallel to the local Mach wave direction such that

$$\left(\frac{d\delta_0}{dx}\right)_{\hat{x}_0} = \frac{1}{M_1} = \frac{3^{\frac{1}{2}}}{2}(\gamma+1)^{\frac{1}{4}}\left(\frac{\gamma-1}{2}\right)^{\omega/4}\Lambda_0^{\frac{1}{4}}M_1^{\omega/2}R_1^{-\frac{1}{4}}\left(\frac{L}{\hat{x}_0}\right)^{\frac{1}{4}} \tag{28}$$

The "interaction distance" $\hat{x}_0$ can, therefore, be given as follows:

$$\frac{\hat{x}_0}{L} = \frac{9}{16}(\gamma+1)\left(\frac{\gamma-1}{2}\right)^{\omega}\Lambda_0\ \frac{M_1^{2\omega+4}}{R_1} \tag{28a}$$

The approximate law of the decay of the shock wave can also be expressed in terms of $\frac{p_2}{p_1}$ as a function of x. By Eq. 18, it is seen that, for $x \leq \hat{x}_0$,

$$\left(\frac{\delta}{L}\right)^2\frac{d(p_2/p_1)}{d(x/L)} = -\left(\frac{\gamma-1}{2}\right)^{\omega}\gamma\Lambda_0\ \frac{M_1^{2(\omega+1)}}{R_1} = \text{constant} \tag{29}$$

Moreover, for the present approximation, Eq. 16b becomes simply

$$\frac{d}{d(x/L)}\left(\frac{p_2}{p_1}\frac{\delta^2}{L^2}\right) = \left(\frac{\gamma-1}{2}\right)^{\omega}2\gamma\Lambda_0\ M_1^{2(\omega+1)}\frac{1}{R_1} \tag{30}$$

Combining Eqs. 29 and 30 yields

$$\frac{p_2}{p_1} = \frac{3\gamma}{2(\gamma+1)^{\frac{1}{2}}}\left(\frac{\gamma-1}{2}\right)^{\omega/2}\Lambda_0^{\frac{1}{2}}\ \frac{M_1^{\omega+2}}{R_1^{\frac{1}{2}}}\ \frac{1}{\left(\frac{x}{L}\right)^{\frac{1}{2}}} \tag{31}$$

Therefore, the shock wave pressure jump is varying inversely to $x^{\frac{1}{2}}$. As the distance from the plate leading edge increases, the shock wave must become weaker. Theoretically, when $x = \hat{x}_0$, the shock wave angle becomes the Mach wave angle and the pressure ratio $\frac{p_2}{p_1}$ should become unity. Substitution of $\frac{\hat{x}_0}{L}$ from Eq. 28a in Eq. 31 gives, however, the result that $\left(\frac{p_2}{p_1}\right)_{\hat{x}_0} = \frac{2\gamma}{\gamma+1} = 1.17$. This conclusion is not surprising because our approximate equation, Eq. 20a, is strictly compatible only with the approximate shock wave condition that $\frac{p_2}{p_1} = \frac{2\gamma}{\gamma+1}\left(M_1 \frac{d\delta}{dx}\right)^2$.

V. CORRECTION FUNCTIONS BY LINEARIZED EQUATIONS

The approximate solutions in the preceding section are based on the approximate system of equations, Eqs. 20a and 22. It is of interest to show in what range of values of x, the approximations are valid. This, indeed, can be demonstrated by the following considerations. By the approximate results of the last section, we have

$$-\frac{\gamma-1}{\gamma+1}\frac{\eta_0}{M_1^2} = O\left[\left(\frac{\mu_w}{\rho_1 u_1}\Lambda_0\right)^{1/2}\frac{x^{3/2}}{M_1^2}\right]$$

$$\frac{\frac{\mu_w}{\rho_1 u_1}}{[1-F(\Lambda_0)]^2}\int_0^x [1-F(\Lambda_0)]^2\left[\frac{(\gamma-1)\left(2-\frac{5\Lambda_0}{6}\right)}{1-F(\Lambda_0)}+\gamma\Lambda_0\right]dx = O\left(\frac{\mu_w}{\rho_1 u_1}\Lambda_0 x\right)$$

Therefore, the approximate equation, Eq. 20a, implies that

$$\frac{x^{1/2}}{M_1^2} << \left(\frac{\mu_w}{\rho_1 u_1}\Lambda_0\right)^{1/2}$$

In other words, the approximate solutions are valid in such range of x's that

$$\frac{x}{L} << \frac{M_1^{2(\omega+2)}\Lambda_0}{R_1} \qquad (32)$$

Hence, as $x \to \hat{x}_0$, (Cf. Eq. 28a), the approximate solution will be in error. However, results in the last section do show that the error involved will not be serious. An indication of this fact is that the value of $\frac{p_2}{p_1}$ at $\hat{x}_0$ on the basis of the approximate results deviates only about 17% from its theoretically true value.

To provide a remedy, we must consider the exact system of equations, Eqs. 20 and 22. As a pair of corrections to the approximate results, we may introduce the functions $\Delta\eta$ and $\Delta\Lambda$ such that they satisfy the following equations:

$$\eta = \eta_0 + \Delta\eta \tag{33}$$

$$\Lambda = \Lambda_0 + \Delta\Lambda \tag{34}$$

$$\frac{\gamma}{2(\gamma+1)}\left(\frac{d\eta}{dx}\right)^2 - \frac{\gamma-1}{\gamma+1}\frac{\eta_0}{M_1^2} = \frac{\mu_w}{\rho_1 u_1}\frac{1}{[1-F(\Lambda)]^2}\int_0^x [1-F(\Lambda)]^2\left[\frac{(\gamma-1)(2-\frac{5}{6}\Lambda)}{1-F(\Lambda)}+\gamma\Lambda\right]dx \tag{35}$$

$$\Lambda = -\frac{\gamma}{\gamma+1}\frac{M_1^2 p_1}{\mu_w u_1}\frac{d\eta}{dx}\left[\frac{d^2\eta}{dx^2} - \frac{1}{2\eta}\left(\frac{d\eta}{dx}\right)^2\right] \tag{36}$$

Assume that the correction functions are small such that the second and higher degree terms in $\Delta\eta$ and $\Delta\Lambda$ and their derivatives can be neglected. Thus, we arrive at a pair of dual relations linear in $\Delta\eta$ and $\Delta\Lambda$ as follows:

$$\begin{aligned}
&\frac{\gamma}{2(\gamma+1)}\left[\left(\frac{d\eta_0}{dx}\right)^2 + 2\left(\frac{d\eta_0}{dx}\right)\frac{d(\Delta\eta)}{dx}\right] - \frac{\gamma-1}{\gamma+1}\frac{\eta_0}{M_1^2} \\
&= \frac{\mu_w}{\rho_1 u_1}\left\{\left[\frac{(\gamma-1)(2-\frac{5\Lambda_0}{6})}{1-F(\Lambda_0)}+\gamma\Lambda_0\right]x(1+2\alpha\Delta\Lambda) + \left[\gamma(1-2\alpha\Lambda_0)+\frac{(\alpha+\beta)(\gamma-1)(2-\frac{5\Lambda_0}{6})}{1-F(\Lambda_0)}\right]\int_0^x \Delta\Lambda\,dx\right\}
\end{aligned} \tag{37}$$

$$\begin{aligned}
\Lambda_0 + \Delta\Lambda = -\frac{\gamma}{\gamma+1}\frac{M_1^2 p_1}{\mu_w u_1}\Bigg\{&\frac{d\eta_0}{dx}\left[\frac{d^2\eta_0}{dx^2} - \frac{1}{2\eta_0}\left(\frac{d\eta_0}{dx}\right)^2\right] + \frac{d\eta_0}{dx}\frac{d^2(\Delta\eta)}{dx^2} \\
&+\left[\frac{d^2\eta_0}{dx^2} - \frac{3}{2\eta_0}\left(\frac{d\eta_0}{dx}\right)^2\right]\frac{d(\Delta\eta)}{dx} + \frac{1}{2\eta_0^2}\left(\frac{d\eta_0}{dx}\right)^3(\Delta\eta)\Bigg\}
\end{aligned} \tag{38}$$

where

$$\alpha = \frac{(dF/d\Lambda)_{\Lambda_0}}{1-F(\Lambda_0)}, \qquad \beta = \frac{5}{6(\gamma-1)(2-\frac{5\Lambda_0}{6})} \tag{39}$$

By the results in the preceding section, the nonlinear parts of Eqs. 37

and 38 are satisfied by Λ_o and η_o defined in Eqs. 24 and 25, i.e.,

$$\frac{\gamma}{2(\gamma+1)}\left(\frac{d\eta_o}{dx}\right)^2 = \frac{\mu_w}{\rho_1 u_1}\left[\frac{(\gamma-1)(2-\frac{5\Lambda_o}{6})}{1-F(\Lambda_o)} + \gamma\Lambda_o\right] x$$

$$\Lambda_o = -\frac{\gamma}{\gamma+1}\,\frac{M_1^2 p_1}{\mu_w u_1}\,\frac{d\eta_o}{dx}\left[\frac{d^2\eta_o}{dx^2} - \frac{1}{2\eta_o}\left(\frac{d\eta_o}{dx}\right)^2\right]$$

Hence, we have from Eqs. 37 and 38

$$\frac{\gamma}{\gamma+1}\left(\frac{d\eta_o}{dx}\right)\frac{d(\Delta\eta)}{dx} - \frac{\gamma-1}{\gamma+1}\,\frac{\eta_o}{M_1^2}$$
$$= \frac{\mu_w}{\rho_1 u_1}\left\{2\alpha\left[\frac{(\gamma-1)(2-\frac{5\Lambda_o}{6})}{1-F(\Lambda_o)} + \gamma\Lambda_o\right] x(\Delta\Lambda) + \left[\gamma(1-2\alpha\Lambda_o) + \frac{(\alpha+\beta)(\gamma-1)(2-\frac{5\Lambda_o}{6})}{1-F(\Lambda_o)}\right]\int_0^x \Delta\Lambda\, dx\right\} \tag{37a}$$

$$\Delta\Lambda = -\frac{\gamma}{\gamma+1}\,\frac{M_1^2 p_1}{\mu_w u_1}\left\{\frac{d\eta_o}{dx}\,\frac{d^2(\Delta\eta)}{dx^2} + \left[\frac{d^2\eta_o}{dx^2} - \frac{3}{2\eta_o}\left(\frac{d\eta_o}{dx}\right)^2\right]\frac{d(\Delta\eta)}{dx} + \frac{1}{2\eta_o^2}\left(\frac{d\eta_o}{dx}\right)^3(\Delta\eta)\right\} \tag{38a}$$

Eqs. 37a and 38a are the pair of linearized dual relations defining the correction functions. Substituting Λ_o and η_o in these equations and eliminating $\Delta\Lambda$ yield the following equation for $\Delta\eta$:

$$a x^{1/2}\frac{d(\Delta\eta)}{dx} + \frac{2}{3} b\, x^{3/2} + c\left[x^{3/2}\frac{d^2(\Delta\eta)}{dx^2} - \frac{7}{4}x^{1/2}\frac{d(\Delta\eta)}{dx} + \frac{9}{8x^{1/2}}(\Delta\eta)\right]$$
$$+ d\int_0^x\left[x_1^{1/2}\frac{d^2(\Delta\eta)}{dx_1^2} - \frac{7}{4x_1^{1/2}}\frac{d(\Delta\eta)}{dx_1} + \frac{9}{8x_1^{3/2}}(\Delta\eta)\right]dx_1 = 0 \tag{40}$$

where $\quad a = \frac{\gamma}{\gamma+1} \qquad b = -\frac{\gamma-1}{\gamma+1}\frac{1}{M_1^2}$

$$c = \frac{4\alpha\gamma\Lambda_o}{\gamma+1} \qquad d = \frac{1}{\gamma+1}\left[\gamma(1-2\alpha\Lambda_o) + (\alpha+\beta)\gamma\Lambda_o\right] \tag{41}$$

Differentiation of Eq. 40 yields

$$c x^3\frac{d^3(\Delta\eta)}{dx^3} + \left(a+d-\frac{c}{4}\right)x^2\frac{d^2(\Delta\eta)}{dx^2} + \left(\frac{a}{2}+\frac{c}{4}-\frac{7}{4}d\right)x\frac{d(\Delta\eta)}{dx} + \left(\frac{9}{8}d-\frac{9}{16}c\right)(\Delta\eta) + b x^2 = 0 \tag{40a}$$

Equation 40a is of the Euler linear equation type. Its complete general solution is

$$\Delta\eta = C_1 x^{a_1} + C_2 x^{a_2} + C_3 x^{a_3} + \frac{b}{\frac{9}{16}c - 3a + \frac{3}{8}d} x^2 \tag{42}$$

where a_1, a_2, and a_3 satisfy the relation

$$F(D) = cD^3 + (a + d - \tfrac{13}{4}c)D^2 + (-\tfrac{1}{2}a - \tfrac{11}{4}d + \tfrac{5}{2}c)D + 9(\tfrac{d}{8} - \tfrac{c}{16}) = (D - a_1)(D - a_2)(D - a_3)$$

and C_1, C_2, and C_3 are arbitrary constants. For the present problem, we can take $C_1 = C_2 = C_3 = 0$ so that the correction function $\Delta\eta$ finally becomes

$$\Delta\eta = \frac{8(\gamma - 1)\, x^2}{3\gamma[7 - (\beta + 5\alpha)\Lambda_0]\, M_1^2} \tag{43}$$

Substituting this last result in Eq. 38a gives the other correction function $\Delta\Lambda$ as follows:

$$\Delta\Lambda = \frac{2(\gamma - 1)}{\gamma + 1} \frac{p_1}{\mu_w u_1} \frac{\left[(\gamma + 1)\Lambda_0 \frac{\mu_w}{\rho_1 u_1}\right]^{1/2}}{[7 - (\beta + 5\alpha)\Lambda_0]} x^{1/2} \tag{44}$$

Hence, we have

$$\eta = x^2 \left\{ \frac{4}{3}\left[(\gamma+1)\left(\frac{\gamma-1}{2}\right)^{\omega} \Lambda_0 \frac{M_1^{2\omega}}{R_1}\right]^{1/2} \left(\frac{x}{L}\right)^{-1/2} + \frac{8(\gamma - 1)}{3\gamma[7 - (\beta + 5\alpha)\Lambda_0] M_1^2} \right\} \tag{45}$$

$$\Lambda = \Lambda_0 + \frac{2(\gamma - 1)}{\gamma M_1^2}[7 - (\beta + 5\alpha)\Lambda_0]^{-1}\left[\frac{\Lambda_0}{\gamma + 1} \frac{R_1}{\left(\frac{\gamma - 1}{2} M_1^2\right)^{\omega}} \frac{x}{L}\right]^{1/2} \tag{46}$$

Evidently, for $x/L \ll M_1^{2(\omega+2)} \Lambda_0 R_1^{-1}$, the second term in each of the above equations, Eqs. 45 and 46, represents a small correction as expected. The importance of the correction terms gradually increases as

$$\frac{x}{L} \to M_1^{2(2+\omega)} \Lambda_0 R_1^{-1}. \qquad \text{For} \qquad \frac{x}{L} >> M_1^{2(2+\omega)} \Lambda_0 R_1^{-1} = O\left(\frac{\hat{x}_0}{L}\right),$$

the correction terms actually dominate; to be sure, for such cases, the above linearization process should fail. Nevertheless, it is interesting to observe that even as $x \to \infty$, we get, from Eq. 45, $\frac{\delta}{x} = O(\frac{1}{M_1})$ which is consistent with the definition of the hypersonic shock regime (Cf. Introduction).

VI. VALUE OF Λ_0

Before we carry on the computations any further, we shall attempt to determine the value of Λ_0 first. In order to do this, the function $F(\Lambda)$ defined in Eq. 17 must be computed for different Λ's. Evaluation of the integral $F(\Lambda)$ can be performed analytically with the following results:

$$\begin{aligned}(1-\tfrac{\Lambda}{6})F(\Lambda) &= \frac{A_1}{2}\log\left|\frac{1+\bar{a}+\bar{b}}{\bar{b}}\right| + \left(A_2 - \frac{A_1\bar{a}}{2}\right)\frac{2}{(4\bar{b}-\bar{a}^2)^{1/2}}\left[\tan^{-1}\frac{2+\bar{a}}{(4\bar{b}-\bar{a}^2)^{1/2}} - \tan^{-1}\frac{\bar{a}}{(4\bar{b}-\bar{a}^2)^{1/2}}\right] \\ &+ \frac{A_3}{2}\log\left|\frac{1+a'+b'}{b'}\right| + \left(A_4 - \frac{A_3 a'}{2}\right)\frac{2}{(4b'-a'^2)^{1/2}}\left[\tan^{-1}\frac{2+a'}{(4b'-a'^2)^{1/2}} - \tan^{-1}\frac{a'}{(4b'-a'^2)^{1/2}}\right] \\ &\text{if} \quad 4\bar{b} > \bar{a}^2, \quad 4b' > a'^2. \end{aligned} \tag{47}$$

$$\begin{aligned}(1-\tfrac{\Lambda}{6})F(\Lambda) &= \frac{A_1}{2}\log\left|\frac{1+\bar{a}+\bar{b}}{\bar{b}}\right| + \left(A_2 - \frac{A_1\bar{a}}{2}\right)\frac{1}{(\bar{a}^2-4\bar{b})^{1/2}}\left[\log\left|\frac{2+\bar{a}-(\bar{a}^2-4\bar{b})^{1/2}}{2+\bar{a}+(\bar{a}^2-4\bar{b})^{1/2}}\right| - \log\left|\frac{\bar{a}-(\bar{a}^2-4\bar{b})^{1/2}}{\bar{a}+(\bar{a}^2-4\bar{b})^{1/2}}\right|\right] \\ &+ \frac{A_3}{2}\log\left|\frac{1+a'+b'}{b'}\right| + \left(A_4 - \frac{A_3 a'}{2}\right)\frac{1}{(a'^2-4b')^{1/2}}\left[\log\left|\frac{2+a'-(a'^2-4b')^{1/2}}{2+a'+(a'^2-4b')^{1/2}}\right| - \log\left|\frac{a'-(a'^2-4b')^{1/2}}{a'+(a'^2-4b')^{1/2}}\right|\right] \\ &\text{if} \quad 4\bar{b} < \bar{a}^2, \quad 4b' < a'^2. \end{aligned} \tag{48}$$

In Eqs. 47 and 48, the following symbols are used:

$$\bar{a} = \frac{1}{2}(b_1 - t) \qquad \bar{b} = \frac{1}{2}y_1 - \left(\frac{b_1}{2}y_1 - b_3\right)\frac{1}{t}$$

$$a' = \frac{1}{2}(b_1 + t) \qquad b' = \frac{1}{2}y_1 + \left(\frac{b_1}{2}y_1 - b_3\right)\frac{1}{t}$$

$$A_1 = -A_3 \qquad A_2 = \frac{1}{b'} - \frac{\bar{b}}{b'}A_4$$

$$A_3 = (\bar{a} - a')^{-1}\left[\left(\frac{\bar{b}}{b'} - 1\right)A_4 - \frac{1}{b'}\right] \qquad A_4 = \left[a'(\bar{a} - a') - (\bar{b} - b')\right]\left[(\bar{a} - a')(\bar{b}a' - \bar{a}b') - (\bar{b} - b')^2\right]^{-1}$$

$$b_1 = \left(\frac{\Lambda}{2} - 2\right)\left(1 - \frac{\Lambda}{6}\right)^{-1}, \quad b_2 = -\frac{\Lambda}{2}\left(1 - \frac{\Lambda}{6}\right)^{-1}, \quad b_3 = \left(2 + \frac{\Lambda}{6}\right)\left(1 - \frac{\Lambda}{6}\right)^{-1}, \quad b_4 = \left(1 - \frac{\Lambda}{6}\right)^{-1}$$

$$t^2 = b_1^2 - 4b_2 + 4y_1 \tag{49}$$

y_1 must satisfy the cubic equation as follows:

$$y_1^3 - b_2 y_1^2 + (b_1 b_3 - 4b_4)y_1 - \left[b_3^2 + b_4(b_1^2 - 4b_2)\right] = 0 \tag{50}$$

for which the discriminant Δ is

$$\Delta = -4p^3 - 27q^2 \tag{51}$$

where

$$p = -\frac{b_2^2}{3} + (b_1 b_3 - 4b_4)$$

$$q = -\frac{2}{27}b_2^3 + \frac{1}{3}b_1 b_2 b_3 - b_3^2 - b_4 b_1^2 + \frac{8}{3}b_2 b_4$$

According to the sign of Δ , y_1 can be given as follows (Ref. 10):

$$\begin{aligned} \Delta < 0, &\quad y_1 = \frac{b_2}{3} + \left[-\frac{q}{2} + \left(-\frac{\Delta}{108}\right)^{1/2}\right]^{1/3} + \left[-\frac{q}{2} - \left(-\frac{\Delta}{108}\right)^{1/2}\right]^{1/3} \\ \Delta > 0, &\quad y_1 = \frac{b_2}{3} + 2\left(-\frac{p}{3}\right)^{1/2}\cos\frac{\phi}{3} \end{aligned} \tag{52}$$

where $\phi = \cos^{-1}\left[\frac{-q/2}{(-p^3/27)^{1/2}}\right]$. Table I gives the values of $F(\Lambda)$ for several Λ's on the basis of the above formulae.

TABLE I

VALUES OF $F(\Lambda)$

Λ	$F(\Lambda)$	Λ	$F(\Lambda)$
.2	.6106	.8	.6085
.4	.6101	1.00	.6087
.6	.6092	--	--

These values of $F(\Lambda)$ have also been independently checked by numerical integration methods. On the basis of these values of $F(\Lambda)$, the value of Λ_o satisfying Eq. 24 is found to be

$$\Lambda_o = .91 \tag{53}$$

VII. GROWTH OF THE BOUNDARY LAYER AND DECAY OF THE SHOCK WAVE

On the basis of the result in Eq. 45, the growth of the viscous layer downstream follows the following law:

$$\frac{\delta}{L} = \left(\frac{x}{L}\right)^{3/4}\left(\Sigma + \sigma\left(\frac{x}{L}\right)^{1/2}\right)^{1/2} \tag{54}$$

where, for brevity, the symbols

$$\Sigma = \frac{4}{3}\left[(\gamma+1)\left(\frac{\gamma-1}{2}\right)^{\omega}\Lambda_o \frac{M_1^{2\omega}}{R_1}\right]^{1/2} \tag{55}$$

$$\sigma = \frac{8(\gamma-1)}{3\gamma\left[7-(\beta+5\alpha)\Lambda_o\right]M_1^2} \tag{56}$$

are introduced. The slope of the shock wave can then be given as follows:

$$\frac{d\delta}{dx} = \frac{3}{4}\,\frac{\Sigma^{1/2}}{\left(\frac{x}{L}\right)^{1/4}}\,\frac{\left[1+\frac{4}{3}\frac{\sigma}{\Sigma}\left(\frac{x}{L}\right)^{1/2}\right]}{\left[1+\frac{\sigma}{\Sigma}\left(\frac{x}{L}\right)^{1/2}\right]^{1/2}} \tag{57}$$

On taking $\sigma = 0$, Eqs. 54 and 57 become Eqs. 26a and 27 respectively. For $0 < \frac{\sigma}{\Sigma}\left(\frac{x}{L}\right)^{1/2} < 1$, Eq. 57 can be expanded as the following series:

$$\frac{d\delta}{dx} = \frac{3}{4}\frac{\Sigma^{1/2}}{(x/L)^{1/4}}\left[1 + \frac{5\sigma}{6\Sigma}\left(\frac{x}{L}\right)^{1/2} - \frac{7}{24}\frac{\sigma^2}{\Sigma^2}\frac{x}{L} + \cdots\cdots\right] \tag{57a}$$

Therefore, it can be stated that Eq. 27 underestimates the slope of the shock wave. A new "interaction distance", x_o, can be found by setting $\frac{d\delta}{dx}$ in Eq. 57 equal to $1/M_1$.

$$\frac{x_o}{L} = \frac{81}{16}\Sigma^2 M_1^4\left[2 - 3\sigma M_1^2 + (4 - 3\sigma M_1^2)^{1/2}\right]^{-2} \tag{58}$$

When $\sigma = 0$, we have $\frac{x_o}{L} = \frac{\hat{x}_o}{L}$ as given in Eq. 28a. To obtain the pressure jump function $\frac{p_2}{p_1}$ consistent with Eqs. 54 and 57, one may proceed as follows: For $x \leq x_o$, Eq. 18 can be cast into the nondimensional form:

$$\left(\frac{\delta}{L}\right)^2\frac{d(p_2/p_1)}{d(x/L)} = -\Lambda\frac{\gamma M_1^2}{R_1}\frac{\mu_w}{\mu_1} \tag{18a}$$

To the same approximation as involved in the linearization of Eq. 35, we can rewrite Eq. 19 as follows:

$$\frac{p_2}{p_1}\delta^2 = M_1^2\frac{\mu_w}{\rho_1 u_1}\left\{\left[\frac{(\gamma-1)(2-\frac{5\Lambda_o}{6})}{1-F(\Lambda_o)} + \gamma\Lambda_o\right]x(1+2\alpha\Delta\Lambda) + \left[\gamma(1-2\alpha\Lambda_o) + \frac{(\alpha+\beta)(\gamma-1)(2-\frac{5\Lambda_o}{6})}{1-F(\Lambda_o)}\right]\int_0^x \Delta\Lambda\, dx\right\} \tag{59}$$

Equation 46 can be rewritten as

$$\Lambda = \Lambda_o\left(1 + \frac{\sigma}{\Sigma}\left(\frac{x}{L}\right)^{1/2}\right) \tag{60}$$

i.e., $\Delta\Lambda = \Lambda_o\frac{\sigma}{\Sigma}\left(\frac{x}{L}\right)^{1/2}$. On the basis of the numerical data given in Table I, Eq. 59 can be simply reduced into the following convenient form:

$$\frac{p_2}{p_1}\delta^2 = M_1^2\frac{\mu_w}{\rho_1 u_1}\left[2\gamma x\Lambda_o\left(1 + .839\frac{\sigma}{\Sigma}\left(\frac{x}{L}\right)^{1/2}\right)\right] \tag{59a}$$

In view of the definition of Λ given in Eq. 60, we may write down the following approximate relation from Eq. 59a:

$$\frac{p_2}{p_1}\delta^2 = M_1^2\frac{\mu_w}{\rho_1 u_1}2\gamma\Lambda x \tag{61}$$

whence we have

$$\frac{d}{dx}\left(\frac{p_2}{p_1}\delta^2\right) = M_1^2 \frac{\mu_w}{\rho_1 u_1} 2\gamma \left(\Lambda + x\frac{d\Lambda}{dx}\right) \tag{61a}$$

Combining Eqs. 18a and 61a and reducing yields

$$\frac{p_2}{p_1}\frac{d(\delta/L)^2}{d(x/L)} = 3\gamma\frac{M_1^2}{R_1}\frac{\mu_w}{\mu_1}\Lambda_o\left(1+\frac{4}{3}\frac{\sigma}{\Sigma}\left(\frac{x}{L}\right)^{\frac{1}{2}}\right) \tag{62}$$

By Eqs. 54 and 57, we get the desired result

$$\frac{p_2}{p_1} = \frac{2\gamma M_1^{2+2\omega}}{R_1}\left(\frac{\gamma-1}{2}\right)^{\omega}\frac{\Lambda_o}{\Sigma}\left(\frac{x}{L}\right)^{-\frac{1}{2}} \tag{63}$$

which is formally the same result previously obtained in Eq. 31. To compute $(p_2/p_1)_{x_o}$, we have the simple relation

$$\left(\frac{p_2}{p_1}\right)_{x_o} = \frac{\gamma}{2(\gamma+1)}\left(2-3\sigma M_1^2+(4-3\sigma M_1^2)^{\frac{1}{2}}\right) \tag{64}$$

Notice that $3\sigma M_1^2$ is, by Eq. 56, a function independent of M_1. For $\Lambda_o = .91$, $\gamma = 1.4$, numerical value of $(p_2/p_1)_{x_o}$ computed on the basis of Eq. 64 gives 1.0147. Since this value deviates by only 1.5% from the theoretical value of unity, the correction procedure devised in Section V can be considered as satisfactory. Thus, at $x = x_o$ defined in Eq. 58, the present method yields solutions that fulfill the simultaneous requirements that $\left(\frac{d\delta}{dx}\right)_{x_o} = \frac{1}{M_1}$ and $\left(\frac{p_2}{p_1}\right)_{x_o} = 1.0$.

VIII. AVERAGE SKIN FRICTION COEFFICIENT ON AN INSULATED FLAT PLATE

In Figs. 2 and 3, curves of $\hat{x}_o/L$ and x_o/L have been plotted, according to Eqs. 28a and 58 respectively, against M_1 for $R_1 = 10^6$, 10^7, and 10^8. In computing these values we take $\gamma = 1.4$, $\omega = .768$, and $\Lambda_o = .91$. It is seen from these figures that the "interaction distance" at the lower Mach numbers are of quite insignificant proportion; however, as the Mach number increases, the "interaction distance" grows very rapidly. The rate of growth of the "interaction distance" depends on the Reynolds number of the flat plate; at the lower Reynolds number, this rate grows noticeably faster. Therefore, at a fixed Reynolds number, the shock wave effects are extended farther downstream as the Mach number increases, at extremely high Mach numbers, the shock wave, so to speak, wraps itself on the entire flat plate, $\frac{x_o}{L} \geq 1$. On the other hand, at a fixed Mach number, the viscous effects on the shock wave are much more pronounced, as expected, when the Reynolds number is low. To insure the validity of the initial assumption of continum flow, it is important to impose certain restriction on the relative magnitudes of the Reynolds number and the Mach number such as the criterion of non-slip flow proposed by Tsien (Ref. 11), viz., $\frac{M_1}{R_1^{1/2}} < \frac{1}{100}$.

The hypersonic viscous flow over a flat plate, according to the present theory, can then be pictured as in Fig. 4. Thus, the average skin friction coefficient on the flat plate can be computed by the following formula:

$$C_F = \frac{x_o}{L} C_{F_s} + \left(1 - \frac{x_o}{L}\right) C_{F_v} \tag{65}$$

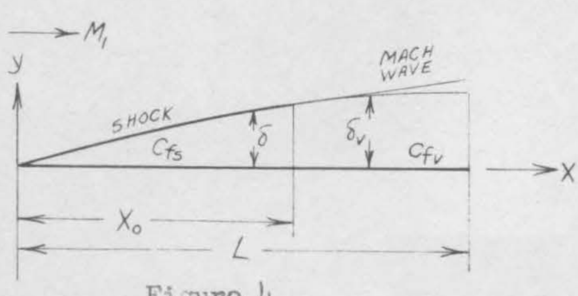

Figure 4

Hypersonic Shock Partially Covers the Flat Plate

where

$$C_{F_s} = \frac{1}{x_o} \int_0^{x_o} C_{f_s} \, dx \tag{66}$$

$$C_{F_v} = \frac{1}{L - x_o} \int_{x_o}^{L} C_{f_v} \, dx \tag{67}$$

where C_{f_s} and C_{f_v} are respectively the local skin friction coefficients in the hypersonic shock flow region ($x \leq x_o$) and the conventional boundary layer flow region ($x_o < x \leq L$)* From the present results, we have

$$C_{f_s} = \frac{\mu_w}{\rho_1 u_1 \delta}\left(2 + \frac{\Lambda}{6}\right) = \left(\frac{\gamma-1}{2}M_1^2\right)^{\omega}\frac{1}{R_1}\left[2 + \Lambda_o\left(1 + \frac{\sigma}{\Sigma}\left(\frac{x}{L}\right)^{1/2}\right)\right]\left\{\Sigma^{1/2}\left(\frac{x}{L}\right)^{3/4}\left(1 + \frac{\sigma}{\Sigma}\left(\frac{x}{L}\right)^{1/2}\right)^{1/2}\right\}^{-1} \tag{68}$$

By von Karman's results (Ref. 12), we take

$$C_{f_v} = \frac{2\mu_w}{\rho_1 u_1 \delta_v} = \frac{1}{2}\sqrt{8f(\theta)}\,\frac{1}{R_1^{1/2}}\left(\frac{\gamma-1}{2}\right)^{\frac{\omega-1}{2}} M_1^{\omega-1}\left(\frac{x}{L}\right)^{-1/2} \tag{69}$$

where $\sqrt{8f(\theta)} = 1.57$. From Eqs. 67 and 69, it is easily obtained that

$$C_{F_v} = 1.57\, R_1^{-1/2}\left(\frac{\gamma-1}{2}\right)^{\frac{\omega-1}{2}} M_1^{\omega-1}\left(1 + \left(\frac{x_o}{L}\right)^{1/2}\right)^{-1} \tag{67a}$$

Combining Eqs. 66 and 68 gives

$$C_{F_s} = \left(\frac{\gamma-1}{2}M_1^2\right)^{\omega} R_1^{-1}\Sigma^{-1/2}\left(\frac{x_o}{L}\right)^{-1/2}\left\{2\int_0^{\left(\frac{x_o}{L}\right)^{1/2}}\frac{d\xi}{\xi^{3/4}\left(1 + \frac{\sigma}{\Sigma}\xi^{1/2}\right)^{1/2}} + \frac{\Lambda_o}{6}\int_0^{\left(\frac{x_o}{L}\right)^{1/2}}\frac{\left(1 + \frac{\sigma}{\Sigma}\xi^{1/2}\right)^{1/2}}{\xi^{3/4}}\,d\xi\right\} \tag{66a}$$

The integrals in Eq. 66a can be evaluated without difficulty; results thus obtained from Eqs. 66a and 67a can be substituted in Eq. 65 to yield finally the following formula:

$$\begin{aligned} C_F = &\left(\frac{\gamma-1}{2}M_1^2\right)^{\omega} R_1^{-1}\Sigma^{-1/2}\left\{\frac{\Lambda_o}{3}\left(\sqrt{\frac{x_o}{L}} + \frac{\sigma}{\Sigma}\frac{x_o}{L}\right)^{1/2} + \left(\frac{\sigma}{\Sigma}\right)^{-1/2}\left(4 + \frac{\Lambda_o}{6}\right)\left[\log\left\{1 + 2\sqrt{\frac{\sigma}{\Sigma}}\left(\sqrt{\frac{\sigma x_o}{\Sigma L}} + \sqrt{\sqrt{\frac{x_o}{L}} + \frac{\sigma}{\Sigma}\frac{x_o}{L}}\right)\right\}\right]\right\} \\ &+ 1.57\left(\frac{\gamma-1}{2}\right)^{\frac{\omega-1}{2}} M_1^{\omega-1} R_1^{-1/2}\left(1 - \sqrt{\frac{x_o}{L}}\right) \end{aligned} \tag{65a}$$

When $\sigma = 0$, Eq. 65a becomes simply

$$C_F = 1.57\left(\frac{\gamma-1}{2}\right)^{\frac{\omega-1}{2}} M_1^{\omega-1} R_1^{-1/2}\left(1 - \sqrt{\frac{\hat{x}_o}{L}}\right) + 2\sqrt{3}\left(\frac{\gamma-1}{2}\right)^{\frac{3\omega}{4}} M_1^{\frac{3\omega}{2}} R_1^{-3/4}\frac{\left(2 + \frac{\Lambda_o}{6}\right)}{(\gamma+1)^{1/4}\Lambda_o^{1/4}}\left(\frac{\hat{x}_o}{L}\right)^{1/4} \tag{70}$$

---

* For convenience sake, it is assumed here that C_{f_s} of the present theory applies also in the leading edge region.

which, by Eq. 28a, can be rewritten into the following convenient form:

$$C_F = 1.57\left(\frac{\gamma-1}{2}\right)^{\frac{\omega-1}{2}} R_1^{-1/2} M_1^{\omega-1} + \left\{ 3\left(2+\frac{\Lambda_o}{6}\right) - \frac{3}{4}\times 1.57\left(\frac{\gamma+1}{\gamma-1}2\Lambda_o\right)^{1/2}\right\}\left(\frac{\gamma-1}{2}\right)^{\omega} R_1^{-1} M_1^{2\omega+1} \Big\} \tag{70a}$$

Without the second term on its right hand side, Eq. 70a simply gives von Kármán's approximate C_F formula. Therefore, the shock wave effects on C_F are to contribute a certain correction function depending on Λ_o, M_1, and R_1 as in Eqs. 65a and 70a. In the above calculations, it is assumed that the hypersonic shock wave is of such a strength that it becomes a Mach wave before arriving at the plate trailing edge. Therefore, Eqs. 65a and 70 applies respectively for the cases $\frac{x_o}{L} \leq 1$ and $\frac{\hat{x}_o}{L} \leq 1$.

Consider, for example, the diagram of x_o/L for the case of $R_1 = 10^7$ in Fig. 3. As M_1 increases beyond 10, x_o/L grows rapidly. Eventually, at $M_1 = 21$, we find $x_o/L = 1$. The condition of continuum flow is violated when M_1 becomes 32. In the range of hypersonic Mach numbers, $21 \leq M_1 \leq 32$, we therefore could anticipate that $x_o/L \geq 1$, i.e., the hypersonic shock tends to wrap itself on the entire length of the flat plate (Fig. 4a). For such cases, the average skin coefficient on the flat plate can be simply given as:

$$C_F = \frac{1}{L}\int_0^L C_{f_s}\, dx \tag{71}$$

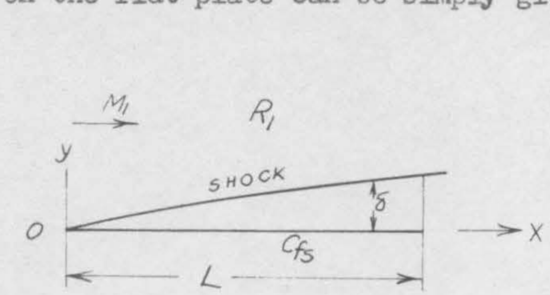

Figure 4a

Hypersonic Shock Wraps Itself on the Entire Plate

Substituting Eq. 68 in Eq. 71 and reducing yields the following formula:

$$C_F = \left(\frac{\gamma-1}{2}M_1^2\right)^{\omega} R_1^{-1} \Sigma^{-1/2} \left\{ \frac{\Lambda_o}{3}\left(1+\frac{\sigma}{\Sigma}\right)^{1/2} + \left(4+\frac{\Lambda_o}{6}\right)\left(\frac{\sigma}{\Sigma}\right)^{-1/2}\left[\log\left\{1+2\sqrt{\frac{\sigma}{\Sigma}}\left(\sqrt{\frac{\sigma}{\Sigma}}+\sqrt{1+\frac{\sigma}{\Sigma}}\right)\right\}\right]\right\} \tag{71a}$$

When $\sigma = 0$, Eq. 71a reduces simply to

$$C_F = 2\sqrt{3}\left(2+\frac{\Lambda_o}{6}\right)\left(\frac{\gamma-1}{2}\right)^{\frac{3\omega}{4}}(\gamma+1)^{-1/4}\Lambda_o^{-1/4}R_1^{-3/4}M_1^{3\omega/2} \tag{71b}$$

Eqs. 71a and 71b apply respectively for the cases $x_o/L \geq 1$ and $\hat{x}_o/L \geq 1$.

IX. DISCUSSIONS

Before we discuss the numerical results, it seems worth while to retrace the development of the present theory in order to bring out its limitations. The hypothesis has been made that the space between the flat plate surface and the shock wave is filled up with a viscous laminar flow which is adequately described by the compressible boundary layer equations. Assuming, in this viscous layer, a plausible velocity distribution function which satisfies the proper boundary conditions and making use of the pressure jump relation at the oblique shock wave front, we derive a pair of dual relations characterizing the nonlinear interactions between the shock wave and the boundary layer. A pair of approximate relations, Eqs. 37 and 38, is then derived in which some important nonlinearities are preserved. Solutions are accordingly obtained on the basis of the approximate system. For a fair evaluation of the significance of the solutions thus found, the following questions must be answered: (1) Are the compressible boundary layer equations valid in the hypersonic viscous regime as postulated above? (2) How are the solutions affected by the choice of the velocity distribution function?

In the introductory section, we have shown that the postulated hypersonic viscous regime can be defined as such that $\frac{\delta}{x} = O(\frac{1}{M_1})$. In the Appendix, we shall show, by the familiar order of magnitude arguments, that Eqs. 1-3 are consistent with this definition of hypersonic regime.

To be specific, we have essentially neglected terms of $O(\frac{1}{M_1^2})$ in these equations. This is, indeed, consistent with the simplification introduced in Eq. 11. Hence, we may conclude that our answer to the first question above is affirmative. In an interesting paper* by Lees and Probestein (Ref. 13), the opinion was, however, expressed that Eqs. 1-3 are probably not valid in the region close to the leading edge where the shock wave coincides with the outer edge of the viscous layer. While we must admit that the present theory breaks down entirely in the leading edge region, our results seem to indicate that the postulated type of hypersonic viscous regime may exist, at very high Mach numbers and a fixed Reynolds number, over a large portion of the plate surface outside the leading edge region (Figs. 2 and 3) where Eqs. 1-3 are expected to be valid.

The second question above is concerning the uniqueness of the solutions. We know that the values of δ, $d\delta/dx$, p_2/p_1 obtained by the present method vary over a range with varying assumptions about the form of u/u_2. Computations using a quadratic velocity profile**, for example, yields a value of Λ_o = .606 which is about 2/3 of the value used in the present paper. However, we found that the functional forms of the solutions are not affected by this choice of the quadratic $\frac{u}{u_2}$-function. Furthermore, we shall point out that although the present analysis is formulated with the explicit condition $\Lambda_o \neq 0$, nevertheless, taking $\Lambda_o = 0$ in Eq. 19, we can obtain an answer which corresponds to Eq. (2.24) in Shen's analysis (Ref. 4). Therefore, it seems appropriate to say that

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* This paper in which an entirely different approach to the same problem has been independently worked out, became available to the authors when the present report was being prepared.

** The boundary conditions $f(1) = 1$, $f(0) = 0$ and $f''(0) = -\Lambda$ are satisfied.

the analytical nature of the solutions is little affected while their numerical values are really affected by the choice of the velocity distribution function. The quartic velocity profile has been chosen for the present calculations because it appears, from past experience with low speed boundary layer flow, that no great improvement in accuracy was gained by considering the velocity profile as a polynomial of higher degree.

It seems, therefore, that the use of the momentum integral method in the manner outlined in the present report, which has the great merit of being comparatively easy, may be expected to give a fairly satisfactory estimate of the interaction effects between the shock wave and the boundary layer at very high Mach numbers. The Mach number range in which the present analysis appears reasonable must be such that (i) $\frac{x_o}{L} = O\left(\frac{M_1^{2\omega+4}}{R_1}\right) > \frac{1}{M_1^2}$ and (ii) $\frac{M_1}{R_1^{1/2}} < \frac{1}{100}$. The first criterion implies that we are only considering the effects of such hypersonic shock waves as have a persistence over a length at least of the order of $1/M_1^2$*. The second criterion is necessary to avoid the use of new boundary conditions pertaining to slip flow.

The above discussions describe the limitations of the present theory. Within the hypersonic Mach number range defined above, we shall calculate the variation of C_F with M_1 for fixed R_1 by the formulae in Eqs. 65a, 70a, 71a, and 71b. The calculations are performed for the cases $R_1 = 10^6$ and 10^7 with $\gamma = 1.4$, $\omega = .768$, $\Lambda_o = .91$. Figs. 5 and 6 show the results thus obtained which are also tabulated in Table II. In these figures, C_F data on the basis of von Kármán's approximate

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* The ϵ- neighborhood of the leading leading mentioned on page 2 has thus been given a specific definition.

formula (Ref. 12) are also given. It is clearly seen that in the hypersonic speed range, the effects of the shock wave on the C_F of the plate can certainly not be neglected. Attention should be drawn to the strong indication that, contrary to the conventional theory, which predicts a steady decrease in C_F as M_1 increases, the present results predict a steady increase of C_F with M_1.

It is very tempting to draw conclusions about the percentage rise of skin friction drag on a flat plate due to the hypersonic shock effects. However, besides the possible inherent inaccuracy due to the assumed velocity profile, our numerical answers almost certainly suffer more or less from the various simplifying assumptions such as (1) $Pr = 1$, (2) $\gamma = 1.4$, and (3) $\omega = .768$. In the large temperature range anticipated under free flight conditions, none of these assumptions are expected to be true and further complications arising from effects of dissociation and ionization may also begin to occur. But, in the test section of a hypersonic wind tunnel where the perfect gas conditions are not greatly violated, these assumptions are satisfactory. Therefore, the numerical data in Figs. 5 and 6 can be regarded as preliminary estimates of the hypersonic shock wave effects. More realistic conclusions of a quantitative nature probably should be deferred until more experimental facts are gathered in the Mach number range of 8 to 15 and higher.

Lees and Probstein (Ref. 13) recently predicted similar increase in the skin friction at hypersonic speeds. Their method of approach and basic assumptions are quite different. The qualitative agreement of the two theories in this respect is rather encouraging.

For $x_o/L \leq 1$, the present theory must admit the coexistence of a hypersonic shock flow regime with an ordinary boundary layer flow regime over the flat plate (Fig. 4). The joining together of these two regimes must be accomplished through a transitional region. The present method is not suitable for exploring this transitional phenomena. Indeed, for the Mach number range corresponding to $x_0/L < 1$, the method of Lees and Probstein's probably yields more dependable results.

For $x_0/L \geqslant 1$, the hypersonic shock regime takes over the entire plate surface. The interaction phenomena between the shock wave and the boundary layer are believed more realistically represented by the present approach. Therefore, for the Mach number range corresponding to $x_0/L \geqslant 1$, the simpler method as presented in the present paper should yield satisfactory results.

Examination of the shock wave effects on the skin friction of a noninsulated flat plate at hypersonic speeds is reserved to a later paper. The corresponding effects on the rate of heat transfer across the plate surface have been shown to be rather small (Ref. 13).

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APPENDIX

COMPRESSIBLE BOUNDARY LAYER EQUATIONS

For the two-dimensional steady flow of a compressible viscous continuum, the basic equations are (i) the Navier Stokes equations:

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial x}\left[-p - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\mu\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] \qquad \text{(A-1)}$$

$$\rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left[-p - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\mu\frac{\partial v}{\partial y}\right] + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] \qquad \text{(A-2)}$$

(ii) the continuity equation:

$$\frac{\partial}{\partial x}\rho u + \frac{\partial}{\partial y}\rho v = 0 \qquad \text{(A-3)}$$

and (iii) the energy equation:

$$\begin{aligned} \rho\left(u\frac{\partial}{\partial x}C_pT + v\frac{\partial}{\partial y}C_pT\right) &= u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) \\ &\quad - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)^2 + 2\mu\left(\frac{\partial u}{\partial x}\right)^2 + 2\mu\left(\frac{\partial v}{\partial y}\right)^2 + \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 \end{aligned} \qquad \text{(A-4)}$$

In these equations, p, ρ, and T satisfy the perfect gas equation:

$$p = R\rho T \qquad \text{(A-5)}$$

μ, k, and T are assumed to be related as follows:

$$\mu \propto T^{\omega} \qquad \text{(A-6)}$$

$$k = \frac{\mu C_p}{P_r} \qquad \text{(A-7)}$$

where C_p and Pr are considered as constants of O(1).

Let a system of reference quantities (with the subscript s) be chosen. Let L represent a suitable length (for instance, the chord of the flat plate). Define:

$$
\begin{aligned}
\xi &= \frac{x}{L} \\
\zeta &= \frac{y}{L} C \\
u^* &= \frac{u}{u_s} \\
v^* &= \frac{v}{u_s} C \\
T^* &= \frac{T}{T_s} A & \mu^* &= \frac{\mu}{\mu_s} E \\
\rho^* &= \frac{\rho}{\rho_s} B & k^* &= \frac{k}{k_s} F \\
p^* &= \frac{p}{p_s} D
\end{aligned}
\tag{A-8}
$$

where A^{-1}, B^{-1}, D^{-1}, E^{-1} and F^{-1} are respectively the temperature ratio, the density ratio, the pressure ratio, the viscosity ratio, and the conductivity ratio, and C is a nondimensional quantity the significance of which will be disclosed presently. We shall regard the nondimensional variables thus defined, viz., ξ , ζ , u*, v*, . . . , as all of the same order of magnitude. They are of O(1). By Eq.(A-5), we have D = O(BA) or D/BA = O(1). By Eqs. (A-6) and (A-7), we have $\frac{F}{A^\omega} = O(1)$, $\frac{E}{A^\omega} = O(1)$.

Nondimensional variables thus defined are introduced into Eqs. (A-1) - (A-4). The results are

<u>1st momentum equation:</u>

$$
\frac{B^{-1}\rho_s u_s^2}{L}\left(\rho^* u^* \frac{\partial u^*}{\partial \xi} + \rho^* v^* \frac{\partial u^*}{\partial \zeta}\right) = -\frac{B^{-1}A^{-1}p_s}{L}\frac{\partial p^*}{\partial \xi} + \frac{C^2 A^{-\omega}\mu_s u_s}{L^2}\frac{\partial}{\partial \zeta}\left(\mu^* \frac{\partial u^*}{\partial \zeta}\right)
$$

$$
\frac{B^{-1}A^{\omega}R_s}{C^2} \times O(1) \quad : \quad \frac{B^{-1}A^{\omega}R_s}{AM_s^2C^2} \times O(1) \quad : \quad O(1)
$$

$$
+ \frac{A^{-\omega}\mu_s u_s}{L^2}\left\{-\frac{2}{3}\frac{\partial}{\partial \xi}\left[\mu^*\left(\frac{\partial u^*}{\partial \xi} + \frac{\partial v^*}{\partial \zeta}\right)\right] + 2\frac{\partial}{\partial \xi}\left(\mu^* \frac{\partial u^*}{\partial \xi}\right) + \frac{\partial}{\partial \zeta}\left(\mu^* \frac{\partial v^*}{\partial \xi}\right)\right\} \tag{A-1a}
$$

$$
: \frac{1}{C^2} \times O(1)
$$

2nd momentum equation:

$$\frac{B^{-1}\rho_s u_s^2}{CL}\left(\rho^* u^* \frac{\partial v^*}{\partial \xi} + \rho^* v^* \frac{\partial v^*}{\partial \zeta}\right) = -\frac{CB^{-1}A^{-1}p_s}{L}\frac{\partial p^*}{\partial \zeta} + \frac{A^{-\omega}\mu_s u_s}{CL^2}\frac{\partial}{\partial \xi}\left(\mu^* \frac{\partial v^*}{\partial \xi}\right)$$

$$\frac{B^{-1}A^{\omega}R_s}{C^4} \times O(1) \quad : \quad \frac{B^{-1}A^{\omega}R_s}{AM_s^2C^2} \times O(1) \quad : \quad \frac{1}{C^4} \times O(1)$$

$$+ \frac{CA^{-\omega}\mu_s u_s}{L^2}\left\{-\frac{2}{3}\frac{\partial}{\partial \zeta}\left[\mu^*\left(\frac{\partial u^*}{\partial \xi} + \frac{\partial v^*}{\partial \zeta}\right)\right] + 2\frac{\partial}{\partial \zeta}\left(\mu^* \frac{\partial v^*}{\partial \zeta}\right) + \frac{\partial}{\partial \xi}\left(\mu^* \frac{\partial u^*}{\partial \zeta}\right)\right\} \tag{A-2a}$$

$$: \frac{1}{C^2} \times O(1)$$

Continuity equation:

$$\frac{B^{-1}\rho_s u_s}{L}\left(\frac{\partial}{\partial \xi}\rho^* u^* + \frac{\partial}{\partial \zeta}\rho^* v^*\right) = 0$$

$$O(1) \tag{A-3a}$$

Energy equation:

$$\frac{B^{-1}A^{-1}T_s u_s \rho_s}{L}\left(\rho^* u^* \frac{\partial}{\partial \xi} C_p T^* + \rho^* v^* \frac{\partial}{\partial \zeta} C_p T^*\right) = \frac{B^{-1}A^{-1}u_s p_s}{L}\left(u^* \frac{\partial p^*}{\partial \xi} + v^* \frac{\partial p^*}{\partial \zeta}\right)$$

$$\frac{B^{-1}A^{\omega}R_s}{AM_s^2C^2} \times O(1) \quad : \quad \frac{B^{-1}A^{\omega}R_s}{AM_s^2C^2} \times O(1)$$

$$+ \frac{k_s A^{-\omega-1}T_s}{L^2}\frac{\partial}{\partial \xi}\left(k^* \frac{\partial T^*}{\partial \xi}\right) + \frac{C^2 A^{-\omega-1}k_s T_s}{L^2}\frac{\partial}{\partial \zeta}\left(k^* \frac{\partial T^*}{\partial \zeta}\right)$$

$$: \frac{1}{AM_s^2C^2} \times O(1) \quad : \quad \frac{1}{AM_s^2} \times O(1)$$

$$+ \frac{\mu_s A^{-\omega}u_s^2}{L^2}\left[-\frac{2}{3}\mu^*\left(\frac{\partial u^*}{\partial \xi} + \frac{\partial v^*}{\partial \zeta}\right)^2 + 2\mu^*\left(\frac{\partial u^*}{\partial \xi}\right)^2 + 2\mu^*\left(\frac{\partial v^*}{\partial \zeta}\right)^2 + 2\mu^* \frac{\partial v^*}{\partial \xi}\frac{\partial u^*}{\partial \zeta}\right]$$

$$: \frac{1}{C^2} \times O(1)$$

$$+ \frac{\mu_s A^{-\omega}u_s^2}{C^2L^2}\mu^*\left(\frac{\partial v^*}{\partial \xi}\right)^2 + \frac{C^2\mu_s A^{-\omega}u_s^2}{L^2}\mu^*\left(\frac{\partial u^*}{\partial \zeta}\right)^2$$

$$: \frac{1}{C^4} \times O(1) \quad : \quad O(1) \tag{A-4a}$$

For brevity, we have written down the relative ratios of the various terms in Eqs. (A-1a) - (A-4a) in terms of their respective order of magnitude. We have also used the symbols $R_s = \frac{\rho_s u_s L}{\mu_s}$, $M_s^2 = \frac{\rho_s u_s^2}{\gamma p_s}$, to denote respectively the reference Reynolds number and the reference Mach number.

For the flat plate problem in hand, the reference quantities can be considered as those pertaining to the free stream. Thus, the subscript s can be replaced by the subscript 1. We shall assume, following the conventional way of thinking, that in a thin layer of the thickness of $O(\delta)$ on the plate surface, the viscous effects can not be neglected. We are to examine the relative importance of the various terms in Eqs. (A-1a) - (A-4a) specifically in the layer where $y = O(\delta)$. By Eq.(A-8), we have then $\frac{y}{x} = O(\frac{\delta}{x}) = O(\frac{1}{C})$. Hence, the nondimensional quantity 1/C characterizes the relative thickness of the boundary layer on the plate surface. The hypersonic criterion that $\frac{\delta}{x} = O(\frac{1}{M_1})$ is therefore equivalent to say that $C = O(M_1)$, where $M_1 >> 1$. In order that the inertia terms and the pressure term in Eq. (A-1a) be of the same order of magnitude, evidently we must have $AM_1^2 = O(1)$ or $A = O(\frac{1}{M_1^2})$. In the thin layer under consideration, we must take the dominant viscous term as being of the same order of magnitude as the inertia terms. Therefore, we must conclude that $B = O(\frac{A^{\omega}R_1}{C^2}) = O(\frac{R_1}{M_1^{2(\omega+1)}})$. With these estimates of the quantities A, B, and C, we can immediately write down the following approximate equations from Eqs. (A-1a) and (A-2a):

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) \tag{A-9}$$

$$\frac{\partial p}{\partial y} = 0 \tag{A-10}$$

In Eqs. (A-9) and (A-10), the terms of $O(\frac{1}{M_1^2})$ and smaller terms are neglected. To the same order of approximation, the energy equation Eq. (A-4a) becomes

$$\rho u \frac{\partial}{\partial x} c_p T + \rho v \frac{\partial}{\partial y} c_p T = u \frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2 \tag{A-11}$$

The continuity equation, of course, retains its original form:

$$\frac{\partial}{\partial x}\rho u + \frac{\partial}{\partial y}\rho v = 0 \qquad \text{(A-3)}$$

These equations are evidently the well-known compressible boundary layer equations. By the above arguments on the order of magnitude, they are shown to be applicable to the hypersonic flow problems characterised by $\frac{\delta}{x} = O\left(\frac{1}{M_1}\right)$. Therefore, they are the basic equations in the present research.

It is interesting to point out that $D = O(BA) = O\left(\frac{R_1}{M_1^{2(\omega+2)}}\right)$, that is, $\frac{p}{p_1} = O\left(\frac{M_1^{2(\omega+2)}}{R_1}\right)$ where p is the static pressure within the viscous layer. However, by Eq. (A-10), we must have $\frac{p}{p_2} = O(1)$ where p_2 is the static pressure behind the leading edge shock wave. On the other hand, the boundary condition pertaining to the oblique shock wave is such that

$$\frac{p_2}{p_1} = \frac{2\gamma}{\gamma+1}\left(M_1\frac{d\delta}{dx}\right)^2 - \frac{\gamma-1}{\gamma+1} = O(1)$$

because by the hypersonic criterion $\left(M_1\frac{d\delta}{dx}\right)^2 = O(1)$. Hence, the present theory should be applicable when $\frac{M_1^{\omega+2}}{R_1^{1/2}} = O(1)$* or when $\frac{x}{L} = O\left(\frac{x_0}{L}\right) = O(1)$ where x_0 is the interaction distance. Therefore, the leading edge region must be regarded as outside the scope of the present investigation.

---

* Taking $\omega = 1$, we have $\frac{M_1^3}{R_1^{1/2}} = O(1)$. $\frac{M_1^3}{R_1^{1/2}}$ was the basic parameter used in the solutions of Lees and Probstein (Ref. 13).

TABLE II

VALUES OF C_F AT VARIOUS M_1 FOR FIXED R_1

$R_1 = 10^6$				$R_1 = 10^7$					
M_1	C_F (Eq.70a)	C_F (Eq.65a)	C_F Von Kármán	M_1	C_F (Eq.70a)	C_F (Eq.65a)	C_F (Eq.71b)	C_F (Eq.71a)	C_F Von Karman
6	.001319	--	.001249						
7	.001309	.0013003	.001205						
8	.001313	.0013019	.001168	8	.000384	--	--	--	.0003694
9	.001334	.0013180	.001138	9	.000379	--	--	--	.0003593
10	.001365	.001345	.001109	10	.000377	.0003742	--	--	.0003506
				12	.000377	.0003737	--	--	.0003363
				16	.000398	.0003921	--	--	.0003145
				20	.000447	.0004353	--	--	.0002986
				22.132	.000484	--	--	.0004783	.0002918
				26	--	--	.000582	.0005719	.0002810
				28	--	--	.000634	.0006303	.0002763
				30	--	--	.000686	.0006831	.0002719
				32	--	--	.000739	.0007365	.0002678

LIST OF FIGURES

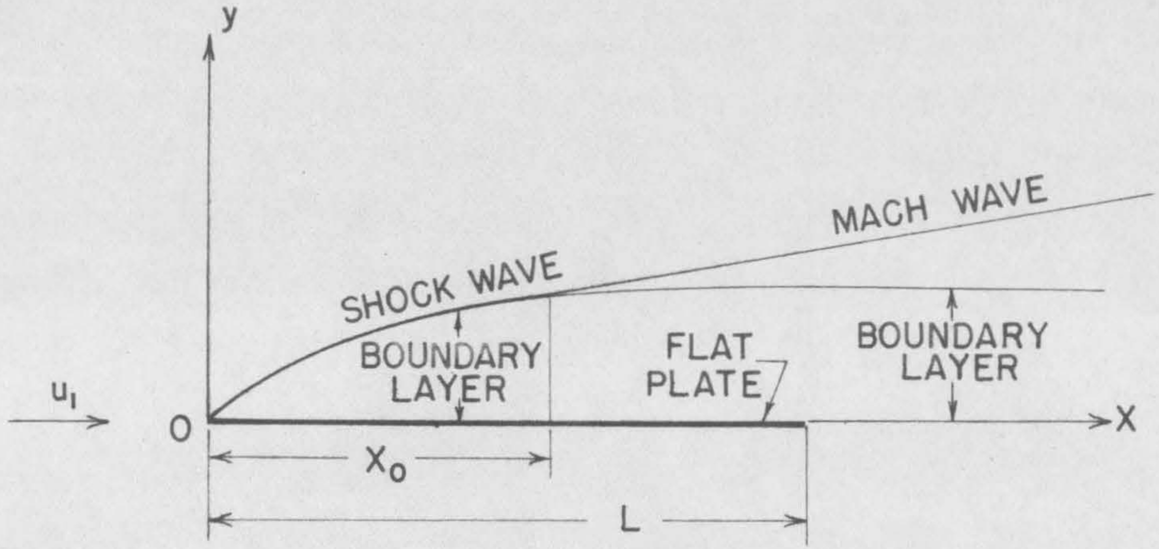

FIG. I — THE COORDINATE SYSTEM AND THE FLAT PLATE

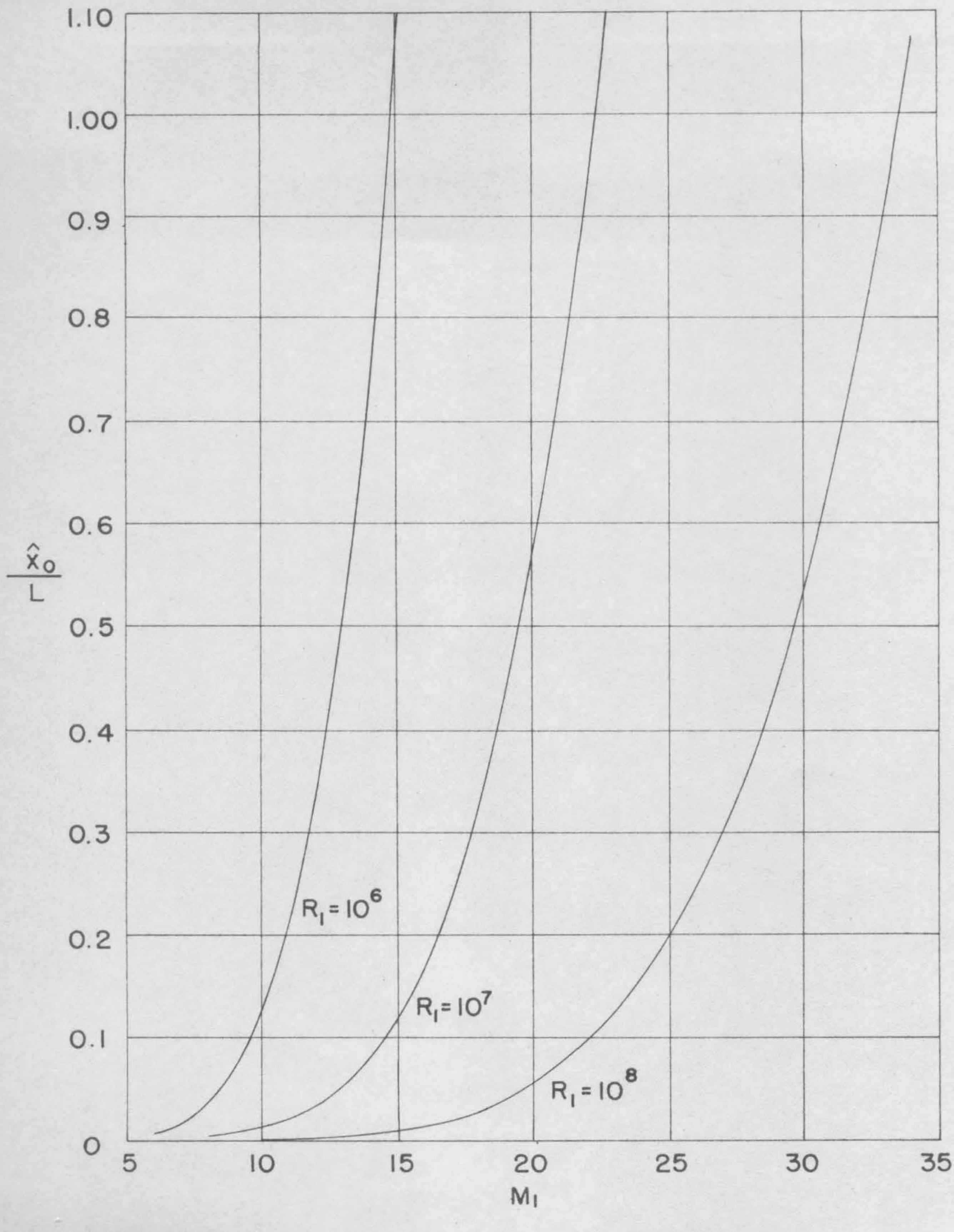

FIG. 2 — INTERACTION DISTANCE RATIO vs MACH NUMBER

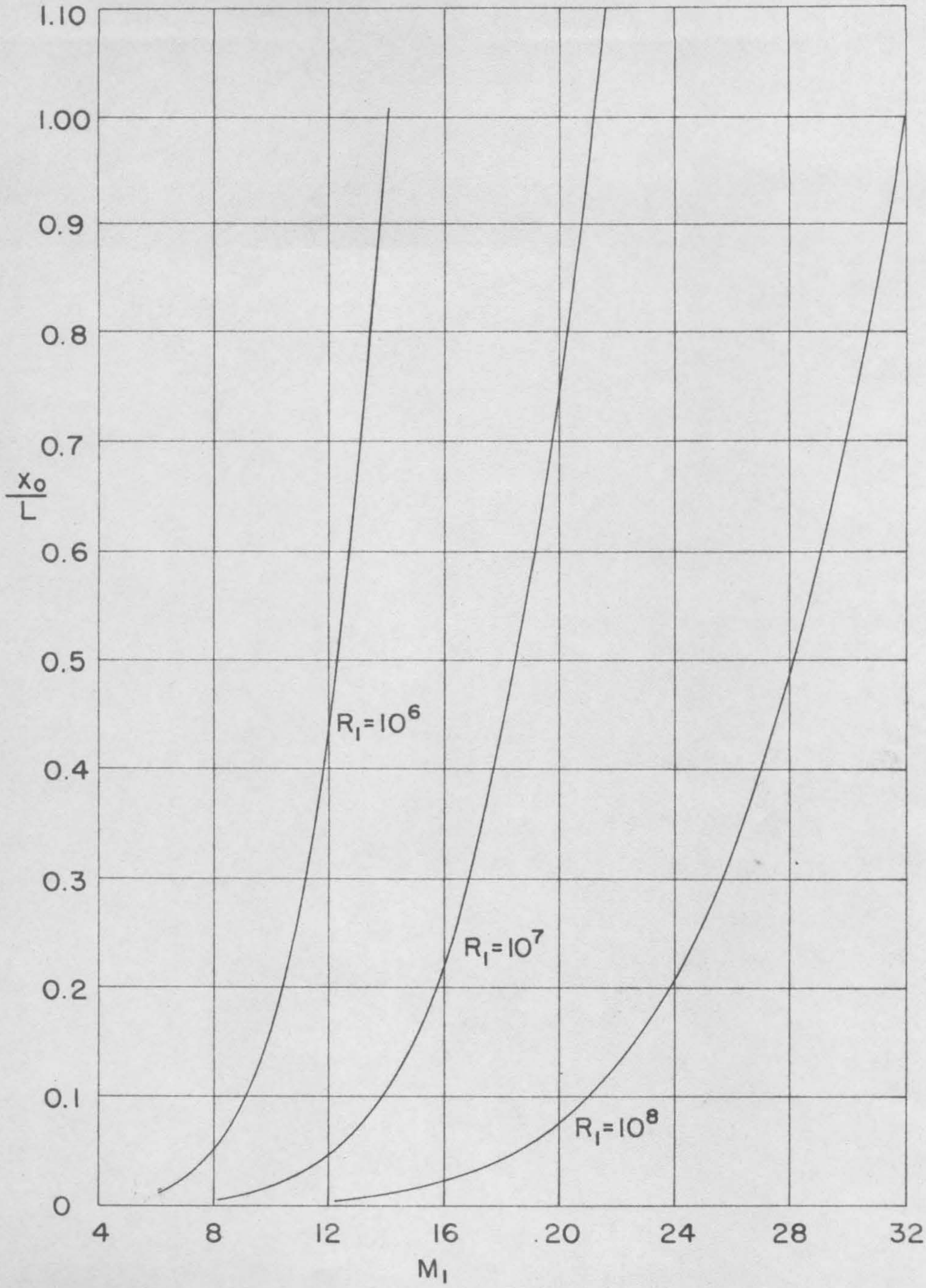

FIG. 3 – INTERACTION DISTANCE RATIO vs MACH NUMBER

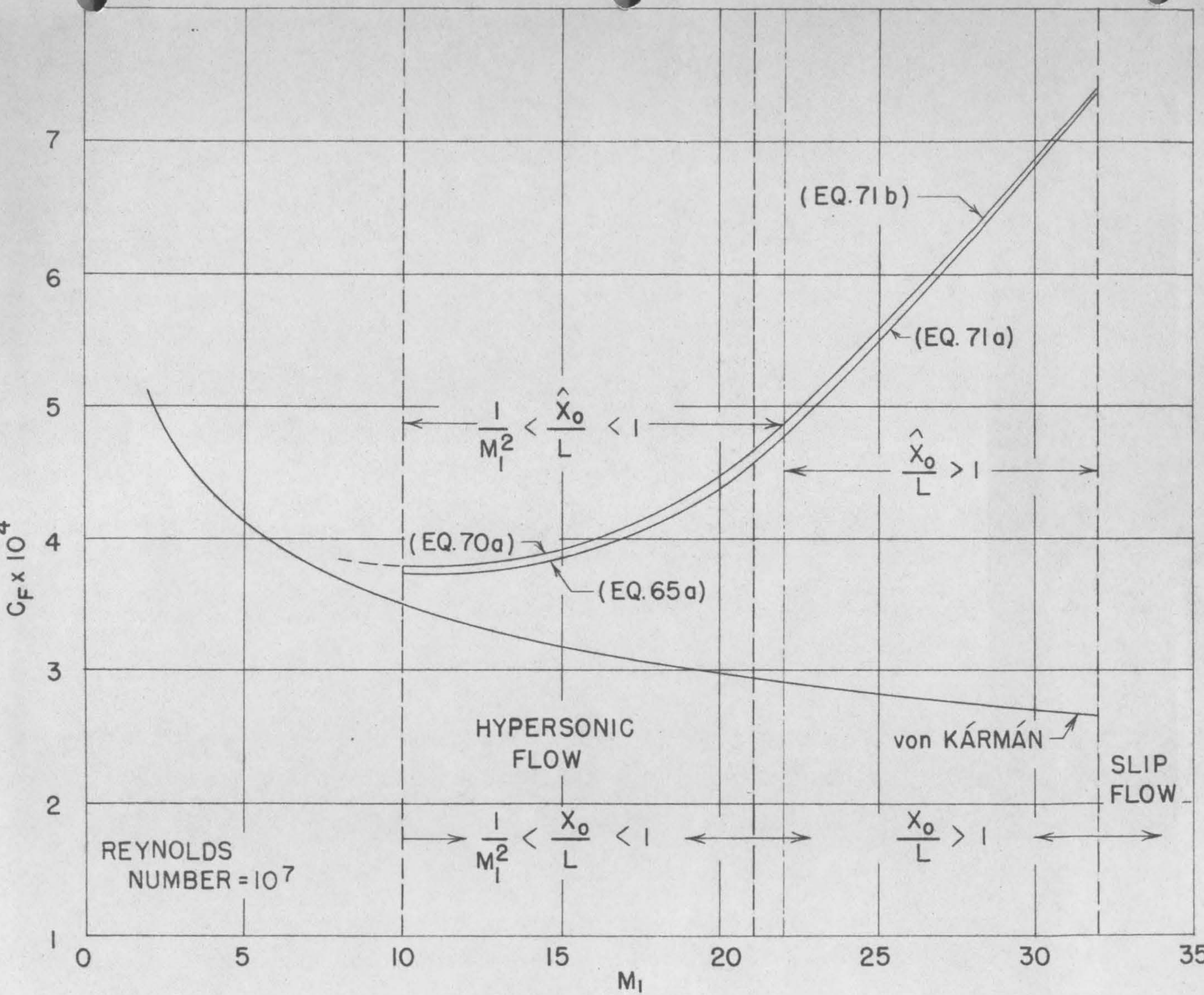

FIG. 5 — AVERAGE SKIN FRICTION COEFFICIENT ON AN INSULATED FLAT PLATE AT HYPERSONIC MACH NUMBERS

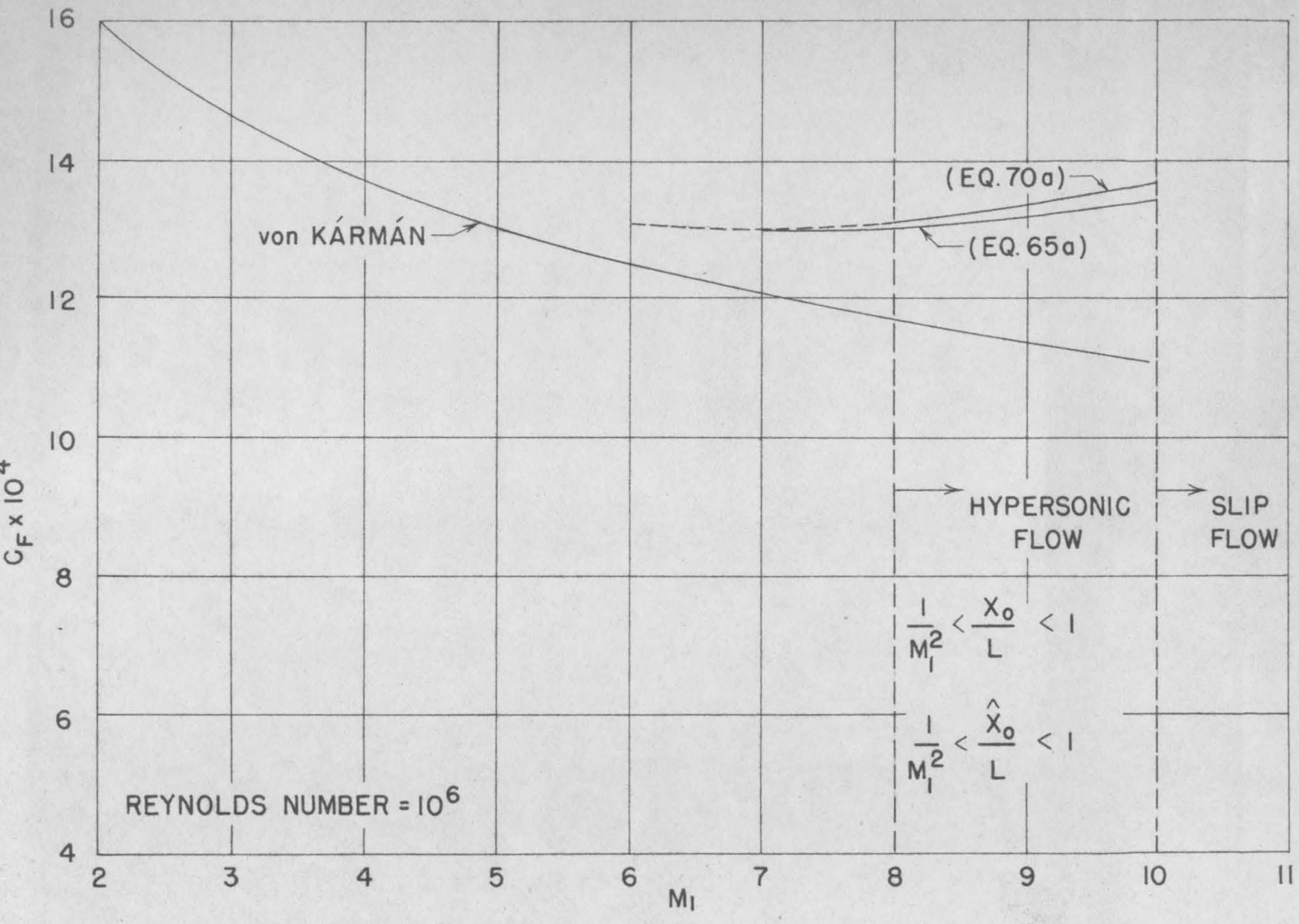

FIG. 6 – AVERAGE SKIN FRICTION COEFFICIENT ON AN INSULATED FLAT PLATE AT HYPERSONIC MACH NUMBERS

Flow, Hypersonic

Contract No. DA-04-495-Ord-19

Army Ordnance Department

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MEMORANDUM NO. 9

ADDITIONS AND ERRATA

SHOCK WAVE EFFECTS ON THE LAMINAR SKIN FRICTION OF AN INSULATED FLAT PLATE AT HYPERSONIC SPEEDS

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Copy No. 23

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September 15, 1952

ERRATA

1. Please replace page 2 with the enclosed page 2 dated 9-15-52.

2. Equation 12 should read (page 5)

$$\frac{u}{u_2} = f(y^*)$$

3. Page 19, line 10, 9th word should read "continuum".

4. Page 20, replace the following equations as indicated:

$$C_{F_s} = \frac{2\mu_w}{\rho_1 u_1 \delta}\left(2 + \frac{\Lambda}{6}\right) = \left(\frac{\gamma-1}{2}M_1^2\right)^{\omega}\frac{2}{R_1}\left[2 + \frac{\Lambda_0}{6}\left(1 + \frac{\sigma}{\Sigma}\left(\frac{x}{L}\right)^{\frac{1}{2}}\right)\right]\left\{\Sigma^{\frac{1}{2}}\left(\frac{x}{L}\right)^{3/4}\left(1 + \frac{\sigma}{\Sigma}\left(\frac{x}{L}\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}\right\}^{-1} \tag{68}$$

$$C_{F_s} = 2\left(\frac{\gamma-1}{2}M_1^2\right)^{\omega}R_1^{-1}\Sigma^{-\frac{1}{2}}\left(\frac{x_0}{L}\right)^{-1}\left\{2\int_0^{\frac{x_0}{L}}\frac{d\xi}{\xi^{3/4}\left(1+\frac{\sigma}{\Sigma}\xi^{\frac{1}{2}}\right)^{\frac{1}{2}}} + \frac{\Lambda_0}{6}\int_0^{\frac{x_0}{L}}\frac{\left(1+\frac{\sigma}{\Sigma}\xi^{\frac{1}{2}}\right)^{\frac{1}{2}}}{\xi^{3/4}}d\xi\right\} \tag{66a}$$

$$\begin{aligned} C_F = {} & 2\left(\frac{\gamma-1}{2}M_1^2\right)^{\omega}R_1^{-1}\Sigma^{-\frac{1}{2}}\left\{\frac{\Lambda_0}{3}\left(\sqrt{\frac{x_0}{L}} + \frac{\sigma}{\Sigma}\frac{x_0}{L}\right)^{\frac{1}{2}} + \left(\frac{\sigma}{\Sigma}\right)^{-\frac{1}{2}}\left(4 + \frac{\Lambda_0}{6}\right)\left[\log\left\{1 + 2\sqrt{\frac{\sigma}{\Sigma}}\left(\sqrt{\frac{\sigma x_0}{\Sigma L}} + \sqrt{\sqrt{\frac{x_0}{L}} + \frac{\sigma x_0}{\Sigma L}}\right)\right\}\right]\right\} \\ & + 1.57\left(\frac{\gamma-1}{2}\right)^{\frac{\omega-1}{2}}M_1^{\omega-1}R_1^{-\frac{1}{2}}\left(1 - \sqrt{\frac{x_0}{L}}\right) \end{aligned} \tag{65a}$$

$$C_F = 1.57\left(\frac{\gamma-1}{2}\right)^{\frac{\omega-1}{2}}M_1^{\omega-1}R_1^{-\frac{1}{2}}\left(1 - \sqrt{\frac{\hat{x}_0}{L}}\right) + 4\sqrt{3}\left(\frac{\gamma-1}{2}\right)^{\frac{3\omega}{4}}M_1^{\frac{3\omega}{2}}R_1^{-3/4}\frac{\left(2 + \frac{\Lambda_0}{6}\right)}{(\gamma+1)^{\frac{1}{4}}\Lambda_0^{\frac{1}{4}}}\left(\frac{\hat{x}_0}{L}\right)^{\frac{1}{4}} \tag{70}$$

5. Page 21, line 12, last word should read "continuum", and the following equations should be replaced as indicated:

$$C_F = 1.57\left(\frac{\gamma-1}{2}\right)^{\frac{\omega-1}{2}}R_1^{-\frac{1}{2}}M_1^{\omega-1} + \left\{6\left(2 + \frac{\Lambda_0}{6}\right) - \frac{3}{4}\times 1.57\left(\frac{\gamma+1}{\gamma-1}2\Lambda_0\right)^{\frac{1}{2}}\right\}\left(\frac{\gamma-1}{2}\right)^{\omega}R_1^{-1}M_1^{2\omega+1} \tag{70a}$$

$$\frac{1}{2}C_F = \left(\frac{\gamma-1}{2}M_1^2\right)^{\omega}R_1^{-1}\Sigma^{-\frac{1}{2}}\left\{\frac{\Lambda_0}{3}\left(1 + \frac{\sigma}{\Sigma}\right)^{\frac{1}{2}} + \left(4 + \frac{\Lambda_0}{6}\right)\left(\frac{\sigma}{\Sigma}\right)^{-\frac{1}{2}}\left[\log\left\{1 + 2\sqrt{\frac{\sigma}{\Sigma}}\left(\sqrt{\frac{\sigma}{\Sigma}} + \sqrt{1 + \frac{\sigma}{\Sigma}}\right)\right\}\right]\right\} \tag{71a}$$

6. Page 22, Equation 71b should read

$$\frac{1}{2}C_F = 2\sqrt{3}\left(2 + \frac{\Lambda_0}{6}\right)\left(\frac{\gamma-1}{2}\right)^{\frac{3\omega}{4}}(\gamma+1)^{-\frac{1}{4}}\Lambda_0^{-\frac{1}{4}}R_1^{-\frac{3}{4}}M_1^{3\omega/2} \tag{71b}$$

7. Please replace pages 33, 38, and 39 with the enclosed pages 33, 38, and 39 dated 9-15-52.

surface that the entire region between the shock wave and the plate surface should be considered as a viscous flow layer. Consequently, for the determination of the shock wave characteristics, viscous effects must be considered (Ref. 4); and for the estimation of the plate skin friction, due care must be exercised to account for the shock wave effects. In the present paper, an approximate theory of the shock wave effects on the hypersonic laminar friction on an insulated flat plate will be presented.

II. BASIC EQUATIONS AND ASSUMPTIONS

The thickness of the shock wave (Refs. 5 and 6) is assumed infinitesimally thin.* The flow variables in the region bounded by the shock wave and the plate surface are assumed to satisfy the compressible boundary layer equations:

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{dp_2}{dx} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) \tag{1}$$

$$\frac{\partial}{\partial x}\rho u + \frac{\partial}{\partial y}\rho v = 0 \tag{2}$$

$$\rho u \frac{\partial}{\partial x}(c_p T) + \rho v \frac{\partial}{\partial y}(c_p T) = u\frac{dp_2}{dx} + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2 \tag{3}$$

The order of magnitude arguments which are responsible for the derivation of the above equations can be carried through in hypersonic flow problems. (Cf. Appendix.) However, these arguments can fail near the base of the shock wave (Ref. 7). In the present analysis, the shock wave is assumed to start from the plate leading edge (Fig. 1). Therefore, the above equations do not hold in the vicinity of the origin, $0 \leq x < \epsilon$. The

---

* The ratio of the shock wave thickness χ and the boundary layer thickness δ can be expressed as (Ref. 5)

$$\frac{\chi}{\delta} = \frac{8\gamma}{\gamma+1}\frac{\mu^*}{\rho^* u_n^* \delta}\frac{(M_{1n}^* + 1)}{(M_{1n}^* - 1)} ,$$

where the superscript * denotes the quantities at conditions pertaining to the critical speed of sound, and the subscript n denotes the normal component to the shock wave front. For the present problem, calculations show that except in the region of the leading edge of the plate, $\frac{\chi}{\delta} << 1$, if $M_{1n}^* - 1 >> M_1^{\frac{2\gamma}{1-\gamma}}$.

TABLE II

VALUES OF C_F AT VARIOUS M_1 FOR FIXED R_1

$R_1 = 10^6$			
M_1	C_F (Eq.70a)	C_F (Eq.65a)	C_F Von Kármán
6	.00149	--	.001249
7	.001569	.001577	.001205
8	.001679	.001690	.001168
9	.001825	.001841	.001138
10	.002009	.002028	.001109

$R_1 = 10^7$					
M_1	C_F (Eq.70a)	C_F (Eq. 65a)	C_F (Eq.71b)	C_F (Eq.71a)	C_F Von Kármán
8	.000420	--	--	--	.000369
9	.000428	--	--	--	.000359
10	.000441	.000443	--	--	.000351
12	.000479	.000482	--	--	.000336
16	.000611	.000617	--	--	.000315
20	.000822	.000831	--	--	.000299
22.132	.000966	--	--	.000957	.000292
26	--	--	.001164	.001144	.000281
28	--	--	.001268	.001261	.000276
30	--	--	.001372	.001366	.000272
32	--	--	.001478	.001473	.000268

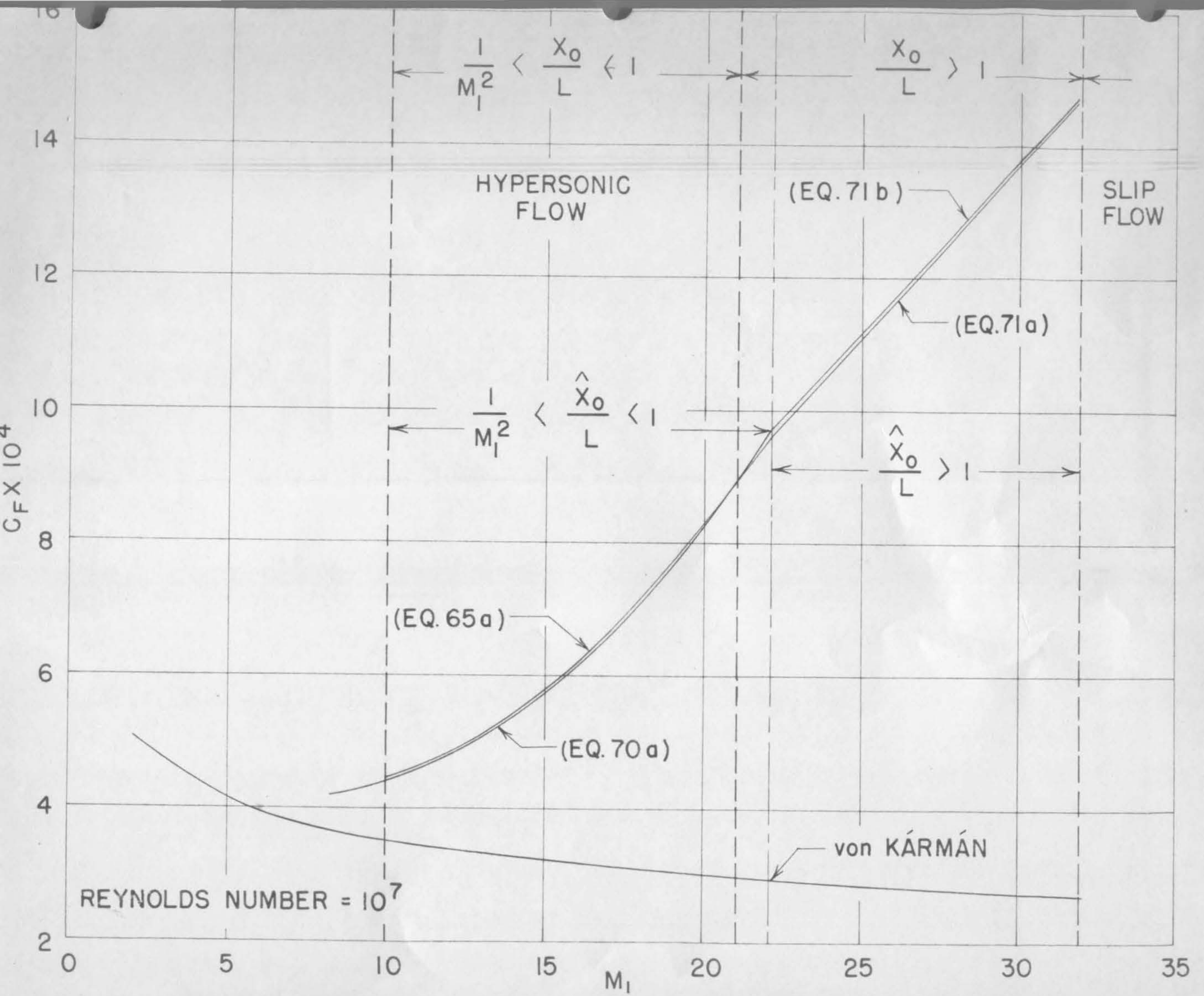

FIG. 5 — AVERAGE SKIN FRICTION COEFFICIENT ON AN INSULATED FLAT PLATE AT HYPERSONIC MACH NUMBERS

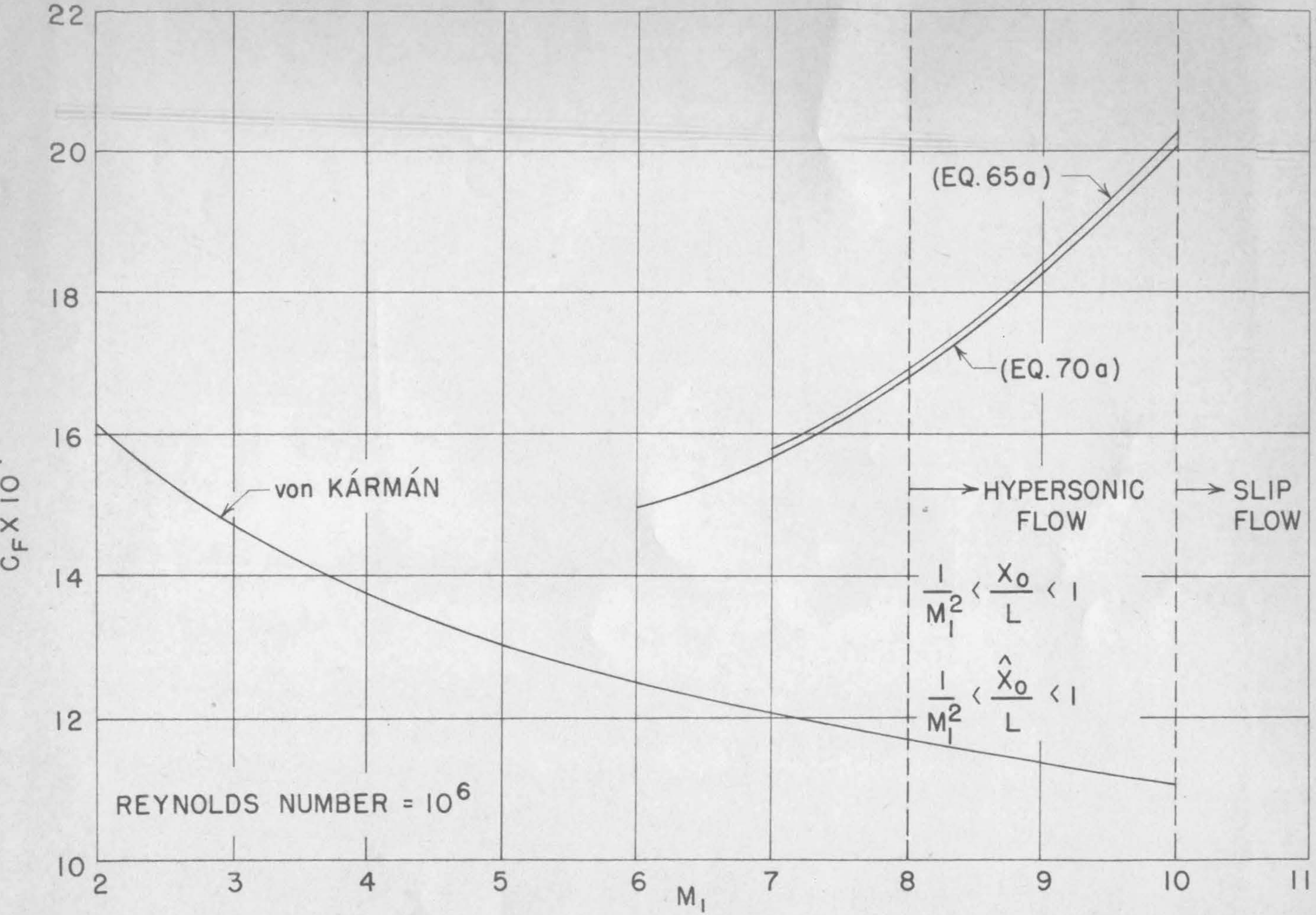

FIG. 6 — AVERAGE SKIN FRICTION COEFFICIENT ON AN INSULATED FLAT PLATE AT HYPERSONIC MACH NUMBERS